On the Polyharmonic Operator with a Periodic Potential

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Abstract

In this paper we obtain the asymptotic formulas of arbitrary order for the Bloch eigenvalues and Bloch functions of the d-dimensional polyharmonic operator $L(l, q(x)) = (-\Delta)^l + q(x)$ with periodic, with respect to arbitrary lattice, potential $q(x)$, where $l \geq 1$ and $d \geq 2$. Then we prove that the number of gaps in the spectrum of the operator $L(l, q(x))$ is finite. In particular, taking $l = 1$, we get the proof of the Bethe -Sommerfeld conjecture for arbitrary dimension and arbitrary lattice.

1 Introduction

In this paper we consider the operator

$$L(l, q(x)) = (-\Delta)^l + q(x),\ x \in \mathbb{R}^d,\ d \geq 2, l \geq 1 \tag{1}$$

with a periodic (relative to a lattice Ω) potential $q(x) \in W_2^s(F)$, where $s \geq s_0 = \frac{3d-1}{2}(3^d + d + 2) + \frac{1}{4}d3^d + d + 6$, $F \equiv \mathbb{R}^d/\Omega$ is a fundamental domain of Ω. Without loss of generality it can be assumed that the measure $\mu(F)$ of F is 1 and $\int_F q(x)dx = 0$. Let $L_t(l, q(x))$ be the operator generated in F by (1) and the conditions:

$$u(x + \omega) = e^{i(t,\omega)}u(x),\ \forall \omega \in \Omega, \tag{2}$$

where $t \in F^\star \equiv \mathbb{R}^d/\Gamma$ and Γ is the lattice dual to Ω, that is, Γ is the set of all vectors $\gamma \in \mathbb{R}^d$ satisfying $(\gamma, \omega) \in 2\pi Z$ for all $\omega \in \Omega$. It is well-known that the spectrum of the operator $L_t(l, q(x))$ consists of the eigenvalues $\Lambda_1(t) \leq \Lambda_2(t) \leq ....$The function $\Lambda_n(t)$ is called n-th band function and its range $A_n = \{\Lambda_n(t) : t \in F^*\}$ is called the n-th band of the spectrum $Spec(L)$ of L and $Spec(L) = \cup_{n=1}^{\infty} A_n$. The eigenfunction $\Psi_{n,t}(x)$ of $L_t(l, q(x))$ corresponding to the eigenvalue $\Lambda_n(t)$ is known as Bloch functions. In the case $q(x) = 0$ these eigenvalues and eigenfunctions are $\mid \gamma + t \mid^{2l}$ and $e^{i(\gamma+t,x)}$ for $\gamma \in \Gamma$.

This paper consists of 4 section. First section is the introduction, where we describe briefly the scheme of this paper and discuss the related papers.

Let the potential $q(x)$ be a trigonometric polynomial

$$\sum_{\gamma\in Q} q_\gamma e^{i(\gamma,x)},$$

where $q_\gamma = (q(x), e^{i(\gamma,x)}) = \int_F q(x)e^{-i(\gamma_1,x)}dx$, and $Q = \{\gamma \in \Gamma : q_\gamma \neq 0\}$ consists of a finite number of vectors γ from Γ. Then the eigenvalue $|\gamma+t|^{2l}$ is called a non-resonance eigenvalue if $\gamma+t$ does not belong to any of the sets

$W^l_{b,\alpha_1} = \{x \in \mathbb{R}^d :\mid\mid x \mid^{2l} - \mid x+b \mid^{2l}\mid < \mid x \mid^{\alpha_1}\}$, that is, if $\gamma+t$ lies far from the diffraction hyperplanes $D_b = \{x \in \mathbb{R}^d :\mid x \mid^2 = \mid x+b \mid^2\}$, where $\alpha_1 \in (0,1)$, $b \in Q$. The idea of the definition of the non-resonance eigenvalue $|\gamma+t|^{2l}$ is the following. If $\gamma+t \notin W^l_{b,\alpha_1}$ then the influence of $q_b e^{i(b,x)}$ to the eigenvalue $|\gamma+t|^{2l}$ is not significant. If $\gamma+t$ does not belong to any of the sets W^l_{b,α_1} for $b \in Q$ then the influence of the trigonometric polynomial $q(x)$ to the eigenvalue $|\gamma+t|^{2l}$ is not significant. Therefore the corresponding eigenvalue of the operator $L_t(l, q(x))$ is close to the eigenvalue $\mid \gamma+t \mid^{2l}$ of $L_t(l,0)$.

If $q(x) \in W_2^s(F)$, then to describe the non-resonance and resonance eigenvalues $|\gamma+t|^{2l}$ of the order of ρ^{2l} (written as $|\gamma+t| \sim \rho$) for big parameter ρ we write the potential $q(x) \in W_2^s(F)$ in the form

$$q(x) = \sum_{\gamma_1\in\Gamma(\rho^\alpha)} q_{\gamma_1} e^{i(\gamma_1,x)} + O(\rho^{-p\alpha}), \tag{3}$$

where $\Gamma(\rho^\alpha) = \{\gamma \in \Gamma : 0 < \mid \gamma \mid < \rho^\alpha)\}$, $p = s-d$, $\alpha = \frac{1}{m}$, $m = 3^d + d + 2$, and the relation $|\gamma+t| \sim \rho$ means that $c_1\rho < |\gamma+t| < c_2\rho$. Here and in subsequent relations we denote by c_i $(i = 1, 2, ...)$ the positive constants, independent on ρ, whose exact values are inessential. Note that $q(x) \in W_2^s(F)$ means that $\sum_\gamma \mid q_\gamma \mid^2 (1+ \mid \gamma \mid^{2s}) < \infty$. If $s \geq d$, then

$$\sum_\gamma \mid q_\gamma \mid < c_3, \quad \sup_{x\in[0,1]} \mid \sum_{\gamma\notin\Gamma(c_1\rho^\alpha)} q_\gamma e^{i(\gamma,x)} \mid \leq \sum_{|\gamma|\geq c_1\rho^\alpha} \mid q_\gamma \mid = O(\rho^{-p\alpha}), \tag{4}$$

i.e., (3) holds. It follows from (4) that the influence of $\sum_{\gamma\notin\Gamma(c_1\rho^\alpha)} q_\gamma e^{i(\gamma,x)}$ to the eigenvalue $|\gamma+t|^{2l}$ is $O(\rho^{-p\alpha})$. If $\gamma+t$ does not belong to any of the sets

$W^l_{b,\alpha_1}(c_2) = \{x \in \mathbb{R}^d :\mid\mid x \mid^{2l} - \mid x+b \mid^{2l}\mid < c_2 \mid x \mid^{\alpha_1}\}$ for $b \in \Gamma(c_1\rho^\alpha)$, then the influence of the trigonometric polynomial $P(x) = \sum_{\gamma\in\Gamma(c_1\rho^\alpha)} q_\gamma e^{i(\gamma,x)}$ to the eigenvalue $|\gamma+t|^{2l}$ is not significant. Thus the corresponding eigenvalue of the operator $L_t(l, q(x))$ is close to the eigenvalue $\mid \gamma+t \mid^{2l}$ of $L_t(l,0)$. Note that changing the values of c_1 and c_2 in the definitions of $W^l_{b,\alpha_1}(c_2)$ and $P(x)$ we obtain the different definitions of the non-resonance eigenvalues. However, in any case we obtain the same asymptotic formulas and the same perturbation theory, that is, this changing does not change anything for asymptotic formulas.

Therefore we can define the non-resonance eigenvalue in different way. In accordance with the case of the trigonometric polynomial it is natural to say that the eigenvalue $|\gamma+t|^{2l}$ is a non-resonance eigenvalue if $\gamma+t$ does not belong to any of the sets $W^l_{b,\alpha_1}(c_2)$ for $\mid b \mid< c_1 p|\gamma+t|^{\alpha}$. However, for simplicity, we give the definitions as follows. By definition, put $\alpha_k = 3^k\alpha$ for $k = 1, 2, ...$ and introduce the sets

$$V^l_{\gamma_1}(\rho^{\alpha_1}) \equiv \{x\in\mathbb{R}^d :\mid\mid x\mid^{2l} - \mid x+\gamma_1\mid^{2l}\mid<\rho^{\alpha_1}\}\cap(R(\tfrac{3}{2}\rho)\backslash R(\tfrac{1}{2}\rho))$$

$$E^l_1(\rho^{\alpha_1},p)\equiv\bigcup_{\gamma_1\in\Gamma(p\rho^{\alpha})}V^l_{\gamma_1}(\rho^{\alpha_1}),\ U^l(\rho^{\alpha_1},p)\equiv(R(\frac{3}{2}\rho)\backslash R(\frac{1}{2}\rho))\backslash E^l_1(\rho^{\alpha_1},p),$$

$$E^l_k(\rho^{\alpha_k},p)\equiv\bigcup_{\gamma_1,\gamma_2,...,\gamma_k\in\Gamma(p\rho^{\alpha})}(\cap^k_{i=1}V^l_{\gamma_i}(\rho^{\alpha_k})),$$

where $R(\rho)=\{x\in\mathbb{R}^d :\mid x\mid<\rho\}$, ρ is a big parameter and the intersection $\cap^k_{i=1}V^l_{\gamma_i}$ in the definition of E^l_k is taken over $\gamma_1,\gamma_2,...,\gamma_k$, that are linearly independent. The set $U^l(\rho^{\alpha_1},p)$ is said to be a non-resonance domain and the eigenvalue $|\gamma+t|^{2l}$ is called a non-resonance eigenvalue if $\gamma+t\in U^l(\rho^{\alpha_1},p)$. The domains $V^l_{\gamma_1}(\rho^{\alpha_1})$ for $\gamma_1\in\Gamma(p\rho^{\alpha})$ are called resonance domains and $\mid\gamma+t\mid^{2l}$ is called a resonance eigenvalue if $\gamma+t\in V^l_{\gamma_1}(\rho^{\alpha_1})$. In Remark 1 we will discuss the relations between sets $W^l_{b,\alpha_1}(c_2)$ and $V^l_b(\rho^{\alpha_1})$.

In section 2 we prove that for each $\gamma+t\in U^l(\rho^{\alpha_1},p)$ there exists an eigenvalue $\Lambda_N(t)$ of the operator $L_t(l,q(x))$ satisfying the following formulae

$$\Lambda_N(t)=\mid\gamma+t\mid^{2l}+F_{k-1}(\gamma+t)+O(\mid\gamma+t\mid^{-3k\alpha}) \tag{5}$$

for $k=1,2,...,[\frac{1}{3}(p-\frac{1}{2}m(d-1))]$, where $[a]$ denotes the integer part of a, $F_0=0$, and F_{k-1} (for $k>1$) is explicitly expressed by the potential $q(x)$ and eigenvalues of $L_t(0)$. Besides, we prove that if the conditions

$$\mid\Lambda_N(t)-\mid\gamma+t\mid^{2l}\mid<\frac{1}{2}\rho^{\alpha_1}, \tag{6}$$

$$\mid b(N,\gamma)\mid>c_4\rho^{-c\alpha} \tag{7}$$

hold, where c is a positive constant,

$$b(N,\gamma)=(\Psi_{N,t},e^{i(\gamma+t,x)}), \tag{8}$$

$\Psi_{N,t}(x)$ is a normalized eigenfunction of $L_t(l,q(x))$ corresponding to $\Lambda_N(t)$, then the following statements are valid:

(a) if $\gamma+t$ is in the non-resonance domain, then $\Lambda_N(t)$ satisfies (5) for $k=1,2,...,[\frac{1}{3}(p-c)]$ (see Theorem 1);

(b) if $\gamma+t\in E^l_s\backslash E^l_{s+1}$, where $s=1,2,...,d-1$, then

$$\Lambda_N(t)=\lambda_j(\gamma+t)+O(\mid\gamma+t\mid^{-k\alpha}), \tag{9}$$

where λ_j is an eigenvalue of a matrix $C(\gamma+t)$ (see (27) and Theorem 2). Moreover, we prove that every big eigenvalue of the operator $L_t(l,q(x))$ for all values of t satisfies one of these formulae.

For investigation of the Bloch function in the non-resonance domain, in section 3, we find the values of quasimomenta $\gamma+t$ for which the corresponding eigenvalues are simple , namely we construct the subset B of $U^1(\rho^{\alpha_1},p)$ with the following properties:

Pr.1. If $\gamma+t\in B$, then there exists a unique eigenvalue, denoted by

$\Lambda(\gamma+t)$, of the operator $L_t(l,q(x))$ satisfying (5). This is a simple eigenvalue of $L_t(l,q(x))$. Therefore we call the set B the simple set of quasimomenta.

Pr.2. The eigenfunction $\Psi_{N(\gamma+t)}(x)\equiv\Psi_{\gamma+t}(x)$ corresponding to the eigenvalue $\Lambda(\gamma+t)$ is close to $e^{i(\gamma+t,x)}$, namely

$$\Psi_N(x)=e^{i(\gamma+t,x)}+O(\mid\gamma+t\mid^{-\alpha_1}), \tag{10}$$

$$\Psi_{\gamma+t}(x)=e^{i(\gamma+t,x)}+\Phi_{k-1}(x)+O(\mid\gamma+t\mid^{-k\alpha_1}),\ k=1,2,...\ , \tag{11}$$

where Φ_{k-1} is explicitly expressed by $q(x)$ and the eigenvalues of $L_t(l,0)$.

Pr.3. The set B has asymptotically full measure on $\mathbb{R}^d$ and contains the intervals $\{a+sb: s\in[-1,1]\}$ such that $\Lambda(a-b)<\rho^{2l}$, $\Lambda(a+b)>\rho^{2l}$, and $\Lambda(\gamma+t)$ is continuous on these intervals. Hence there exists $\gamma+t$ such that $\Lambda(\gamma+t)=\rho^{2l}$. It implies the validity of Bethe-Sommerfeld conjecture for $L(l,q(x))$. These results is proved in section 4.

Construction of the set B consists of two steps.

Step 1. We prove that all eigenvalues $\Lambda_N(t)\sim\rho^{2l}$ of the operator $L_t(l,q(x))$ lie in the $\varepsilon_1=\rho^{-d-2\alpha}$ neighborhood of the numbers

$F(\gamma+t)=\mid\gamma+t\mid^{2l}+F_{k_1-1}(\gamma+t)$, $\lambda_j(\gamma+t)$ (see (5), (9)), where $k_1=[\frac{d}{3\alpha}]+2$. We call these numbers as the known parts of the eigenvalues. Moreover, for $\gamma+t\in U^l(\rho^{\alpha_1},p)$ there exists $\Lambda_N(t)$ satisfying $\Lambda_N(t)=F(\gamma+t)+o(\varepsilon_1)$.

Step 2. By eliminating the set of quasimomenta $\gamma+t$, for which the known parts $F(\gamma+t)$ of $\Lambda_N(t)$ are situated from the known parts $F(\gamma^{'}+t)$, $\lambda_j(\gamma^{'}+t)$ $(\gamma^{'}\neq\gamma)$ of other eigenvalues at a distance less than $2\varepsilon_1$, we construct the set B with the following properties: if $\gamma+t\in B$, then the following conditions (called simplicity conditions for $\Lambda_N(t)$) hold

$$\mid F(\gamma+t)-F(\gamma^{'}+t)\mid\geq 2\varepsilon_1 \tag{12}$$

for $\gamma^{'}\in K\backslash\{\gamma\}$, $\gamma^{'}+t\in U^1(\rho^{\alpha_1},p)$ and

$$\mid F(\gamma+t)-\lambda_j(\gamma^{'}+t)\mid\geq 2\varepsilon_1 \tag{13}$$

for $\gamma^{'}\in K,\gamma^{'}+t\in E_k^1\backslash E_{k+1}^1$, $j=1,2,...$, where K is the set of $\gamma^{'}\in\Gamma$ satisfying $\mid F(\gamma+t)-\mid\gamma^{'}+t\mid^{2l}\mid<\frac{1}{3}\rho^{\alpha_1}$. Thus we define the simple set B as follows

Definition 1 *The simple set B is the set of*

$x\in U^1(\rho^{\alpha_1},p)\cap(R(\frac{3}{2}\rho-\rho^{\alpha_1-1})\backslash R(\frac{1}{2}\rho+\rho^{\alpha_1-1}))$ such that $x=\gamma+t$, where $\gamma\in\Gamma,t\in F^\star$, and the simplicity conditions (12), (13) hold.

As a consequence of these conditions the eigenvalue $\Lambda_N(t)$ does not coincide with other eigenvalues. To prove this, namely to prove the Pr.1 and (10), we show that for any normalized eigenfunction $\Psi_N(x)$ corresponding to $\Lambda_N(t)$ the following equality holds:

$$\sum_{\gamma^{'}\in\Gamma\backslash\gamma} \mid b(N,\gamma^{'}) \mid^2 = O(\rho^{-2\alpha_1}). \tag{14}$$

The listed all results (division the eigenvalues $|\gamma+t|^{2l}$, for big $\gamma\in\Gamma$, into two groups: non-resonance ones and resonance ones, the proof of the formulas (5), (9), construction and investigations of the simply set B, the proof of the asymptotic formulas (11) for Bloch function and implication the proof of the Bethe-Sommerfeld conjecture for arbitrary dimension and arbitrary lattices from these formulas) for the first time were obtained in papers [12-14,16] for the Schrodinger operator $L(1,q(x))$. For the first time in [12-14] we constructed the simple set B with the Pr.1 and Pr.3., though in those papers we emphasized the Bethe-Zommerfeld conjecture. Note that for this conjecture and for Pr.1, Pr.3. it is enough to prove that the left-hand side of (14) is less than $\frac{1}{4}$ (we proved this inequality in [12-14] and as noted in Theorem 3 of [13] and in [16] the proof of this inequality does not differ from the proof of (14)). From (10) we got (11) by iteration (see [16]) . The enlarged form of this results is written in [15],[18],[19].

The main difficulty and the crucial point of papers [12-14] were the construction and investigations of the simple set B of quasimomenta in neighborhood of the surface $\{\gamma+t\in U^1(\rho^{\alpha_1},p): F(\gamma+t)=\rho^2\}$. This difficulty of the perturbation theory of $L(1,q(x))$ is of a physical nature and it is connected with the complicated picture of the crystal diffraction. If $d=2,3$, then $F(\gamma+t)=\mid\gamma+t\mid^2$ and the matrix $C(\gamma+t)$ corresponds to the Schrodinger operator with directional potential $q_{\gamma_1}(x)=\sum_{n\in Z} q_{n\gamma_1}e^{i(n\gamma_1,x)}$ (see [13]). So for construction of the simple set B of quasimomenta we eliminated the vicinities of the diffraction planes and the sets connected with directional potential (see (12), (13)). Besides, for nonsmooth potentials $q(x)\in L_2(\mathbb{R}^2/\Omega)$,we eliminated a set, which is described in the terms of the number of states (see [12,17]). The simple sets B of quasimomenta for the first time are constructed and investigated (hence also the main difficulty and the crucial point of perturbation theory of $L(1,q)$ is investigated) in [13] for $d=3$ and in [12,14] for the cases:

1. $d=2$, $q(x)\in L_2(F)$;
2. $d>2$, $q(x)$ is a smooth potential.

Then, Yu.E. Karpeshina proved (see [6],[7],[8]) the convergence of the perturbation series of $L(l,q)$ with a wide class of nonsmooth potentials $q(x)$ for a set, that is similar to B, of quasimomenta in the cases:

1. $2l>d$; 2. $4l>d+1$, $(2l\leq d)$; 3. $d=3, l=1$, and using it she proved the validity of the Bethe-Sommerfeld conjecture in these cases. In papers [2,3] asymptotic formulas for eigenvalues and Bloch function of two and three dimensional operator $L_t(1,q(x))$ were obtained. In [4] asymptotic formulae for non-resonance eigenvalues of $L_0(1,q(x))$ were obtained.

For the first time M.M. Skriganov [10,11] proved the validity of the Bethe-Sommerfeld conjecture for the Schrodinger operator for dimension $d = 2, 3$ for arbitrary lattice, for dimension $d > 3$ for rational lattice, and for the operator $L(l, q(x))$ for $2l > d$. The Skriganov's method is based on the detail investigation of the arithmetic and geometric properties of the lattice. B.E.J.Dahlberg and E.Trubowits [1] using an asymptotic of Bessel function, gave the beautiful proof of this conjecture for the two dimensional Scrodinger operator. B. Helffer and A. Mohamed [5], by investigations the integrated density of states, proved the validity of the Bethe-Sommerfeld conjecture for the Scrodinger operator for $d \leq 4$, for arbitrary lattice. Recently Parnovski and Sobelev [9] proved this conjecture for the operator $L(l, q(x))$, for $8l > d + 3$.

The method of this paper and papers [12-14] is a first and unique, for the present, by which the validity of the Bethe-Sommerfeld conjecture for arbitrary lattice and for arbitrary dimension is proved. For the operator $L(l, q)$, in order to avoid eclipsing the essence by technical details, we assume that $l \geq 1$. It can be replaced by $l > n_{s,d}$, where $n_{s,d} < 1$ and depends on the smoothness s of the potential $q(x) \in W_2^s(\mathbb{R}^d/\Omega)$ and the dimension d .

In this paper for the different types of the measures of the subset A of $\mathbb{R}^d$ we use the same notation $\mu(A)$. By $\mid A \mid$ we denote the number of elements of the set $A \subset \Gamma$ and use the following obvious fact. If $a \sim \rho$, then the number of elements of the set $\{\gamma + t : \gamma \in \Gamma\}$ satisfying $\mid\mid \gamma + t \mid -a \mid < 1$ is less than $c_5 \rho^{d-1}$. Therefore the number of eigenvalues of $L_t(l, q)$ lying in $(a^{2l} - \rho^{2l-1}, a^{2l} + \rho^{2l-1})$ is less than $c_5 \rho^{d-1}$. Besides, we use the inequalities:

$$\begin{aligned} &\alpha_1 + d\alpha < 1 - \alpha\,, \qquad d\alpha < \frac{1}{2}\alpha_d, \quad k_1 \leq \frac{1}{3}(p - \frac{1}{2}(m(d-1)), \\ &\quad p_1\alpha_1 \geq p\alpha, \qquad 3k_1\alpha > d + 2\alpha, \qquad \alpha_k + (k-1)\alpha < 1, \\ &\quad \alpha_{k+1} > 2(\alpha_k + (k-1))\alpha \end{aligned} \tag{15}$$

for $k = 1, 2, ..., d$, which follow from the definitions $p = s - d$, $\alpha_k = 3^k\alpha$, $\alpha = \frac{1}{m}$, $m = 3^d + d + 2$, $k_1 = [\frac{d}{3\alpha}] + 2$, $p_1 = [\frac{p}{3}] + 1$ of the numbers $p, m, \alpha_k, \alpha, k_1, p_1$.

2 Asymptotic Formulae for Eigenvalues

In this section we obtain the asymptotic formulas for the eigenvalues by iteration of the formula

$$(\Lambda_N - \mid \gamma + t \mid^{2l}) b(N, \gamma) = (\Psi_{N,t}(x) q(x), e^{i(\gamma + t, x)}), \tag{16}$$

where $\gamma + t \in U^l(\rho^{\alpha_1}, p)$ and $b(N, \gamma)$ is defined in (8). Introducing into (16) the expansion (3) of $q(x)$, we get

$$(\Lambda_N - \mid \gamma + t \mid^{2l}) b(N, \gamma) = \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} q_{\gamma_1} b(N, \gamma - \gamma_1) + O(\rho^{-p\alpha}). \tag{17}$$

From the relations (16), (17) it follows that

$$b(N,\gamma^{'})=\frac{(\Psi_{N,t}q(x),e^{i(\gamma^{'}+t,x)})}{\Lambda_N-\mid\gamma^{'}+t\mid^{2l}}=\sum_{\gamma_1\in\Gamma(\rho^\alpha)}\frac{q_{\gamma_1}b(N,\gamma^{'}-\gamma_1)}{\Lambda_N-\mid\gamma^{'}+t\mid^{2l}}+O(\rho^{-p\alpha}) \quad (18)$$

for all vectors $\gamma^{'}\in\Gamma$ satisfying the inequality

$$\mid\Lambda_N-\mid\gamma^{'}+t\mid^{2l}\mid>\frac{1}{2}\rho^{\alpha_1}. \quad (19)$$

If (6) holds and $\gamma+t\in U^l(\rho^{\alpha_1},p)$, then

$$\mid\mid\gamma+t\mid^{2l}-\mid\gamma-\gamma_1+t\mid^{2l}\mid>\rho^{\alpha_1},\ \ \mid\Lambda_N-\mid\gamma-\gamma_1+t\mid^{2l}\mid>\frac{1}{2}\rho^{\alpha_1} \quad (20)$$

for all $\gamma_1\in\Gamma(p\rho^\alpha)$. Hence the vector $\gamma-\gamma_1$ for $\gamma+t\in U(\rho^{\alpha_1},p)$ and $\gamma_1\in\Gamma(p\rho^\alpha)$ satisfies (19). Therefore, in (18) one can replace $\gamma^{'}$ by $\gamma-\gamma_1$ and write

$$b(N,\gamma-\gamma_1)=\sum_{\gamma_2\in\Gamma(\rho^\alpha)}\frac{q_{\gamma_2}b(N,\gamma-\gamma_1-\gamma_2)}{\Lambda_N-\mid\gamma-\gamma_1+t\mid^{2l}}+O(\rho^{-p\alpha}).$$

Substituting this for $b(N,\gamma-\gamma_1)$ into the right-hand side of (17) and isolating the terms containing the multiplicand $b(N,\gamma)$, we get

$$(\Lambda_N-\mid\gamma+t\mid^{2l})b(N,\gamma)=\sum_{\gamma_1,\gamma_2\in\Gamma(\rho^\alpha)}\frac{q_{\gamma_1}q_{\gamma_2}b(N,\gamma-\gamma_1-\gamma_2)}{\Lambda_N-\mid\gamma-\gamma_1+t\mid^{2l}}+O(\rho^{-p\alpha})=$$

$$\sum_{\gamma_1\in\Gamma(\rho^\alpha)}\frac{\mid q_{\gamma_1}\mid^2 b(N,\gamma)}{\Lambda_N-\mid\gamma-\gamma_1+t\mid^{2l}}+\sum_{\substack{\gamma_1,\gamma_2\in\Gamma(\rho^\alpha),\\ \gamma_1+\gamma_2\neq0}}\frac{q_{\gamma_1}q_{\gamma_2}b(N,\gamma-\gamma_1-\gamma_2)}{\Lambda_N-\mid\gamma-\gamma_1+t\mid^{2l}}+O(\rho^{-p\alpha}),$$

since $q_{\gamma_1}q_{\gamma_2}=\mid q_{\gamma_1}\mid^2$ for $\gamma_1+\gamma_2=0$ and the last summation is taken under the condition $\gamma_1+\gamma_2\neq0$. Repeating this process $p_1\equiv[\frac{p}{3}]+1$ times, i.e., in the last summation replacing $b(N,\gamma-\gamma_1-\gamma_2)$ by its expression from (18) (in (18) replace $\gamma^{'}$ by $\gamma-\gamma_1-\gamma_2$) and isolating the terms containing $b(N,\gamma)$ etc., we obtain

$$(\Lambda_N-\mid\gamma+t\mid^{2l})b(N,\gamma)=A_{p_1}(\Lambda_N,\gamma+t)b(N,\gamma)+C_{p_1}+O(\rho^{-p\alpha}), \quad (21)$$

where $A_{p_1}(\Lambda_N,\gamma+t)=\sum_{k=1}^{p_1}S_k(\Lambda_N,\gamma+t)$,

$$S_k(\Lambda_N,\gamma+t)=\sum_{\gamma_1,\ldots,\gamma_k\in\Gamma(\rho^\alpha)}\frac{q_{\gamma_1}q_{\gamma_2}...q_{\gamma_k}q_{-\gamma_1-\gamma_2-\ldots-\gamma_k}}{\prod_{j=1}^{k}(\Lambda_N-\mid\gamma+t-\sum_{i=1}^{j}\gamma_i\mid^{2l})},$$

$$C_{p_1}=\sum_{\gamma_1,\ldots,\gamma_{p_1+1}\in\Gamma(\rho^\alpha)}\frac{q_{\gamma_1}q_{\gamma_2}...q_{\gamma_{p_1+1}}b(N,\gamma-\gamma_1-\gamma_2-...-\gamma_{p_1+1})}{\prod_{j=1}^{p_1}(\Lambda_N-\mid\gamma+t-\sum_{i=1}^{j}\gamma_i\mid^{2l})}.$$

Here the summations for S_k and C_{p_1} are taken under the additional conditions $\gamma_1+\gamma_2+...+\gamma_s\neq 0$ for $s=1,2,...,k$ and $s=1,2,...,p_1$ respectively. These conditions and the inclusion $\gamma_i\in\Gamma(\rho^\alpha)$ for $i=1,2,...,p_1$ imply the relation $\sum_{i=1}^{j}\gamma_i\in\Gamma(p\rho^\alpha)$. Therefore from the second inequality in (20) it follows that the absolute values of the denominators of the fractions in S_k and C_{p_1} are greater than $(\frac{1}{2}\rho^{\alpha_1})^k$ and $(\frac{1}{2}\rho^{\alpha_1})^{p_1}$ respectively. Hence the first inequality in (4) and $p_1\alpha_1\geq p\alpha$ (see the fourth inequality in (15)) yield

$$C_{p_1}=O(\rho^{-p_1\alpha_1})=O(\rho^{-p\alpha}),\ S_k(\Lambda_N,\gamma+t)=O(\rho^{-k\alpha_1}),\forall k=1,2,...,p_1. \quad (22)$$

Since we used only the condition (6) for Λ_N, it follows that

$$S_k(a,\gamma+t)=O(\rho^{-k\alpha_1}) \quad (23)$$

for all $a\in\mathbb{R}$ satisfying $\mid a-\mid\gamma+t\mid^{2l}\mid<\frac{1}{2}\rho^{\alpha_1}$. Thus finding N such that Λ_N is close to $\mid\gamma+t\mid^{2l}$ and $b(N,\gamma)$ is not very small, then dividing both sides of (21) by $b(N,\gamma)$, we get the asymptotic formulas for Λ_N.

Theorem 1 *(a) Suppose $\gamma+t\in U^l(\rho^{\alpha_1},p)$. If (6) and (7) hold, then Λ_N satisfies formulas (5) for $k=1,2,...,[\frac{1}{3}(p-c)]$, where*

$$F_s=O(\rho^{-\alpha_1}),\forall s=0,1,..., \quad (24)$$

and $F_0=0$, $F_s=A_s(\mid\gamma+t\mid^{2l}+F_{s-1},\gamma+t)$ for $s=1,2,....$

(b) For $\gamma+t\in U^l(\rho^{\alpha_1},p)$ there exists an eigenvalue Λ_N of $L_t(l,q(x))$ satisfying (5).

Proof. (a) To prove (5) in case $k=1$ we divide both side of (21) by $b(N,\gamma)$ and use (7), (22). Then we obtain

$$\Lambda_N-\mid\gamma+t\mid^{2l}=O(\rho^{-\alpha_1}). \quad (25)$$

This and $\alpha_1=3\alpha$ (see the end of the introduction) imply that formula (5) for $k=1$ holds and $F_0=0$. Hence (24) for $s=0$ is also proved. Moreover, from (23), we obtain $S_k(\mid\gamma+t\mid^{2l}+O(\rho^{-\alpha_1}),\gamma+t)=O(\rho^{-\alpha_1})$ for $k=1,2,....$ Therefore (24) for arbitrary s follows from the definition of F_s by induction. Now we prove (5) by induction on k. Suppose (5) holds for $k=j$, that is,

$\Lambda_N=\mid\gamma+t\mid^{2l}+F_{k-1}(\gamma+t)+O(\rho^{-3k\alpha})$. Substituting this into $A_{p_1}(\Lambda_N,\gamma+t)$ in (21) and dividing both sides of (21) by $b(N,\gamma)$, we get

$$\Lambda_N=\mid\gamma+t\mid^{2l}+A_{p_1}(\mid\gamma+t\mid^{2l}+F_{j-1}+O(\rho^{-j\alpha_1}),\gamma+t)+O(\rho^{-(p-c)\alpha})=$$
$$\mid\gamma+t\mid^{2l}+\{A_{p_1}(\mid\gamma+t\mid^{2l}+F_{j-1}+O(\rho^{-j\alpha_1}),\gamma+t)-$$
$$A_{p_1}(\mid\gamma+t\mid^{2l}+F_{j-1},\gamma+t)\}+A_{p_1}(\mid\gamma+t\mid^{2l}+F_{j-1},\gamma+t)+O(\rho^{-(p-c)\alpha}).$$

To prove (a) for $k=j+1$ we need to show that the expression in curly brackets is equal to $O(\rho^{-(j+1)\alpha_1})$. It can be checked by using (4), (20), (24) and the

obvious relation

$$\frac{1}{\prod_{j=1}^{s}(\mid \gamma+t\mid^{2l}+F_{j-1}+O(\rho^{-j\alpha_1})-\mid \gamma+t-\sum_{i=1}^{s}\gamma_i\mid^{2l})}-$$
$$\frac{1}{\prod_{j=1}^{s}(\mid \gamma+t\mid^{2l}+F_{j-1}-\mid \gamma+t-\sum_{i=1}^{s}\gamma_i\mid^{2l})}$$
$$=\frac{1}{\prod_{j=1}^{s}(\mid \gamma+t\mid^{2l}+F_{j-1}-\mid \gamma+t-\sum_{i=1}^{s}\gamma_i\mid^{2l})}(\frac{1}{1-O(\rho^{-(j+1)\alpha_1})}-1)$$

$=O(\rho^{-(j+1)\alpha_1})$ for $s=1,2,...,p_1$.

(b) Let A be the set of indices N satisfying (6). Using (16) and Bessel inequality, we obtain

$$\sum_{N\notin A}\mid b(N,\gamma)\mid^2=\sum_{N\notin A}\mid\frac{(\Psi_N(x),q(x)e^{i(\gamma+t,x)})}{\Lambda_N-\mid\gamma+t\mid^{2l}}\mid^2=O(\rho^{-2\alpha_1})$$

Hence, by the Parseval equality, we have $\sum_{N\in A}\mid b(N,\gamma)\mid^2=1-O(\rho^{-2\alpha_1})$. This and the inequality $\mid A\mid< c_5\rho^{d-1}=c_5\rho^{(d-1)m\alpha}$ (see the end of the introduction) imply that there exists a number N satisfying $\mid b(N,\gamma)\mid>\frac{1}{2}(c_5)^{-1}\rho^{-\frac{(d-1)m}{2}\alpha}$, that is, (7) holds for $c=\frac{(d-1)m}{2}$. Thus Λ_N satisfies (5) due to (a) ■

Theorem 1 shows that in the non-resonance case the eigenvalue of the perturbed operator $L_t(l,q(x))$ is close to the eigenvalue of the unperturbed operator $L_t(l,0)$. However, in Theorem 2 we prove that if $\gamma+t\in\cap_{i=1}^{k}V_{\gamma_i}^{1}(\rho^{\alpha_k})\backslash E_{k+1}^{1}$ for $k\geq 1$, where $\gamma_1,\gamma_2,...,\gamma_k$ are linearly independent vectors of $\Gamma(p\rho^{\alpha})$, then the corresponding eigenvalue of $L_t(l,q(x))$ is close to the eigenvalue of the matrix constructed as follows. Introduce the sets:

$B_k\equiv B_k(\gamma_1,\gamma_2,...,\gamma_k)=\{b:b=\sum_{i=1}^{k}n_i\gamma_i,n_i\in Z,\mid b\mid<\frac{1}{2}\rho^{\frac{1}{2}\alpha_{k+1}}\}$,

$$B_k(\gamma+t)=\gamma+t+B_k=\{\gamma+t+b:b\in B_k\},\tag{26}$$

$B_k(\gamma+t,p_1)=\{\gamma+t+b+a:b\in B_k,\mid a\mid<p_1\rho^{\alpha},a\in\Gamma\}$.

Denote by h_i+t for $i=1,2,...,b_k$ the vectors of $B_k(\gamma+t,p_1)$, where $b_k\equiv b_k(\gamma_1,\gamma_2,...,\gamma_k)$ is the number of the vectors of $B_k(\gamma+t,p_1)$. Define the matrix $C(\gamma+t,\gamma_1,\gamma_2,...,\gamma_k)\equiv(c_{i,j})$ by the formulas

$$c_{i,i}=\mid h_i+t\mid^{2l},\ c_{i,j}=q_{h_i-h_j},\ \forall i\neq j,\tag{27}$$

where $i,j=1,2,...,b_k$.

Using the mean value theorem it is not hard to see that if $x\in\mathbb{R}^d$, $\mid x\mid\sim\rho$, $\gamma_1\in\Gamma$, $\mid x+\gamma_1\mid\sim\rho$, then

$$\mid x\mid^{2l}-\mid x+\gamma_1\mid^{2l}=a^{2(l-1)}(\mid x\mid^2-\mid x+\gamma_1\mid^2)\tag{28}$$

where $a\sim\rho$. Therefore for $l\geq 1$ and $k=1,2,...$, we have

$$(\cap_{i=1}^{k}V_{\gamma_i}^{l}(\rho^{\alpha_k}))\subset\cap_{i=1}^{k}V_{\gamma_i}^{1}(\rho^{\alpha_k})),\tag{29}$$

$$U^1(\rho^{\alpha_1},p)\subset U^l(\rho^{\alpha_1},p) \tag{30}$$

Taking into account this, we consider the resonance eigenvalue $\mid \gamma+t\mid^{2l}$ for $\gamma+t\in(\cap_{i=1}^{k}V_{\gamma_i}^1(\rho^{\alpha_k}))$ by using the following lemma.

Lemma 1 *If $\gamma+t\in\cap_{i=1}^{k}V_{\gamma_i}^1(\rho^{\alpha_k})\backslash E_{k+1}^1$, $h+t\in B_k(\gamma+t,p_1)$, $(h-\gamma^{'}+t)\notin B_k(\gamma+t,p_1)$, then*

$$\mid\mid \gamma+t\mid^{2l}-\mid h-\gamma^{'}-\gamma_1^{'}-\gamma_2^{'}-...-\gamma_s^{'}+t\mid^{2l}\mid>\frac{1}{5}\rho^{\alpha_{k+1}}, \tag{31}$$

where $\gamma^{'}\in\Gamma(\rho^{\alpha})$, $\gamma_j^{'}\in\Gamma(\rho^{\alpha})$, $j=1,2,...,s$ and $s=0,1,...,p_1-1$.

Proof. The inequality $p>2p_1$ (see the end of the introduction) and the conditions of Lemma 1 imply that
$h-\gamma^{'}-\gamma_1^{'}-\gamma_2^{'}-...-\gamma_s^{'}+t\in B_k(\gamma+t,p)\backslash B_k(\gamma+t)$ for all $s=0,1,...,p_1-1$.
It follows from the definitions of $B_k(\gamma+t,p)$, B_k that (see (26))
$h-\gamma^{'}-\gamma_1^{'}-\gamma_2^{'}-...-\gamma_s^{'}+t=\gamma+t+b+a$, where

$$\mid b\mid<\frac{1}{2}\rho^{\frac{1}{2}\alpha_{k+1}},\mid a\mid<p\rho^{\alpha},\ \gamma+t+b+a\notin\gamma+t+B_k. \tag{32}$$

Then (31) has the form

$$\mid\mid \gamma+t+a+b\mid^{2l}-\mid\gamma+t\mid^{2l}\mid>\frac{1}{5}\rho^{\alpha_{k+1}}. \tag{33}$$

It follows from (28) that, to verify (33) it is enough to prove it for $l=1$. To prove (33) for $l=1$ we consider two cases:

Case 1. $a\in P$, where $P=Span\{\gamma_1,\gamma_2,...,\gamma_k\}$. Since $b\in B_k\subset P$, we have $a+b\in P$. This with the third relation in (32) imply that $a+b\in P\backslash B_k$,i.e., $\mid a+b\mid\geq\frac{1}{2}\ \rho^{\frac{1}{2}\alpha_{k+1}}$. Consider the orthogonal decomposition $\gamma+t=y+v$ of $\gamma+t$, where $v\in P$ and $y\perp P$. First we prove that the projection v of any vector $x\in\cap_{i=1}^{k}V_{\gamma_i}^1(\rho^{\alpha_k})$ on P satisfies

$$\mid v\mid=O(\rho^{(k-1)\alpha+\alpha_k}). \tag{34}$$

For this we turn the coordinate axis so that $Span\{\gamma_1,\gamma_2,...,\gamma_k\}$ coincides with the span of the vectors $e_1=(1,0,0,...,0)$, $e_2=(0,1,0,...,0),...,$ e_k. Then $\gamma_s=\sum_{i=1}^{k}\gamma_{s,i}e_i$ for $s=1,2,...,k$. Therefore the relation $x\in\cap_{i=1}^{k}V_{\gamma_i}^1(\rho^{\alpha_k})$ implies that

$$\sum_{i=1}^{k}\gamma_{s,i}x_i=O(\rho^{\alpha_k}),s=1,2,...,k;\ x_n=\frac{\det(b_{j,i}^n)}{\det(\gamma_{j,i})},\ n=1,2,...,k,$$

where $x=(x_1,x_2,...,x_d),\gamma_j=(\gamma_{j,1},\gamma_{j,2},...,\gamma_{j,k},0,0,...,0)$, $b_{j,i}^n=\gamma_{j,i}$ for $n\neq j$ and $b_{j,i}^n=O(\rho^{\alpha_k})$ for $n=j$. Taking into account that the determinant $\det(\gamma_{j,i})$

is the volume of the parallelepiped $\{\sum_{i=1}^{k} b_i\gamma_i : b_i \in [0,1], i = 1,2,...,k\}$ and using $\mid \gamma_{j,i} \mid< p\rho^{\alpha}$ (since $\gamma_j \in \Gamma(p\rho^{\alpha})$), we get the estimations

$$x_n = O(\rho^{\alpha_k+(k-1)\alpha}) \ ,\forall n = 1,2,...,k;\ \forall x \in \cap_{i=1}^{k} V_{\gamma_i}^1(\rho^{\alpha_k}). \tag{35}$$

Hence (34) holds. Therefore, using the inequalities $\mid a+b \mid \geq \frac{1}{2}\ \rho^{\frac{1}{2}\alpha_{k+1}}$ (see above), $\alpha_{k+1} > 2(\alpha_k + (k-1)\alpha)$ (see the seventh inequality in (15)), and the obvious equalities $(y,v) = (y,a) = (y,b) = 0$,

$$\mid \gamma+t+a+b \mid^2 - \mid \gamma+t \mid^2 = \mid a+b+v \mid^2 - \mid v \mid^2, \tag{36}$$

we obtain the estimation (33).

Case 2. $a \notin P$. First we show that

$$\mid\mid \gamma+t+a \mid^2 - \mid \gamma+t \mid^2\mid \geq \rho^{\alpha_{k+1}}. \tag{37}$$

Suppose, to the contrary, that it does not hold. Then $\gamma+t \in V_a^1(\rho^{\alpha_{k+1}})$. On the other hand $\gamma+t \in \cap_{i=1}^{k} V_{\gamma_i}^1(\rho^{\alpha_{k+1}})$ (see the conditions of Lemma 1). Therefore we have $\gamma+t \in E_{k+1}^1$ which contradicts the conditions of the lemma. Hence (37) is proved. Now, to prove (33) we write the difference $\mid \gamma+t+a+b \mid^2 - \mid \gamma+t \mid^2$ as the sum of $d_1 \equiv\mid \gamma+t+a+b \mid^2 - \mid \gamma+t+b \mid^2$ and $d_2 \equiv\mid \gamma+t+b \mid^2 - \mid \gamma+t \mid^2$. Since $d_1 = \mid \gamma+t+a \mid^2 - \mid \gamma+t \mid^2 + 2(a,b)$, it follows from the inequalities (37), (32) that $\mid d_1 \mid > \frac{2}{3}\ \rho^{\alpha_{k+1}}$. On the other hand, taking $a = 0$ in (36), we have $d_2 = \mid b+v \mid^2 - \mid v \mid^2$. Therefore (34), the first inequality in (32) and the seventh inequality in (15) imply that $\mid d_2 \mid < \frac{1}{3}\ \rho^{\alpha_{k+1}}$, $\mid d_1 \mid - \mid d_2 \mid > \frac{1}{3}\rho^{\alpha_{k+1}}$, that is, (33) holds ■

Theorem 2 (a) *Suppose* $\gamma+t \in (\cap_{i=1}^{k} V_{\gamma_i}^1(\rho^{\alpha_k}))\backslash E_{k+1}^1$, *where* $k = 1,2,...,d-1$. *If (6) and (7) hold, then there is an index* j *such that*

$$\Lambda_N(t) = \lambda_j(\gamma+t) + O(\rho^{-(p-c-\frac{1}{4}d3^d)\alpha}), \tag{38}$$

where $\lambda_1(\gamma+t) \leq \lambda_2(\gamma+t) \leq ... \leq \lambda_{b_k}(\gamma+t)$ *are the eigenvalues of the matrix* $C(\gamma+t,\gamma_1,\gamma_2,...,\gamma_k)$ *defined in (27).*

(b) *Every eigenvalue* $\Lambda_N(t)$ *of the operator* $L_t(l,q(x))$ *satisfies either (5) or (38) for* $c = \frac{m(d-1)}{2}$.

Proof. (a)Writing the equation (17) for all $h_i+t \in B_k(\gamma+t,p_1)$, we obtain

$$(\Lambda_N - \mid h_i+t \mid^{2l})b(N,h_i) = \sum_{\gamma^{'} \in \Gamma(\rho^{\alpha})} q_{\gamma^{'}} b(N,h_i-\gamma^{'}) + O(\rho^{-p\alpha}) \tag{39}$$

for $i = 1,2,...,b_k$ (see (26) for definition of $B_k(\gamma+t,p_1)$). It follows from (6) and Lemma 1 that if $(h_i-\gamma^{'}+t) \notin B_k(\gamma+t,p_1)$, then

$$\mid \Lambda_N - \mid h_i-\gamma^{'}-\gamma_1-\gamma_2-...-\gamma_s+t \mid^{2l}\mid > \frac{1}{6}\rho^{\alpha_{k+1}},$$

where $\gamma^{'} \in \Gamma(\rho^{\alpha}), \gamma_j \in \Gamma(\rho^{\alpha}),\ j = 1, 2, ..., s$ and $s = 0, 1, ..., p_1 - 1$. Therefore, applying the formula (18) p_1 times, using (4) and $p_1\alpha_{k+1} > p_1\alpha_1 \geq p\alpha$ (see the fourth inequality in (15)), we see that if $(h_i - \gamma^{'} + t) \notin B_k(\gamma + t, p_1)$, then

$$b(N, h_i - \gamma^{'}) =$$

$$\sum_{\gamma_1,...,\gamma_{p_1-1}\in\Gamma(\rho^{\alpha})} \frac{q_{\gamma_1}q_{\gamma_2}...q_{\gamma_{p_1}}b(N, h_i - \gamma^{'} - \sum_{i=1}^{p_1}\gamma_i)}{\prod_{j=0}^{p_1-1}(\Lambda_N- \mid h_i - \gamma' + t - \sum_{i=1}^{j}\gamma_i \mid^{2l})} + \tag{40}$$

$$+O(\rho^{-p\alpha}) = O(\rho^{p_1\alpha_{k+1}}) + O(\rho^{-p\alpha}) = O(\rho^{-p\alpha}).$$

Hence (39) has the form

$$(\Lambda_N- \mid h_i + t \mid^{2l})b(N, h_i) = \sum_{\gamma^{'}} q_{\gamma'}b(N, h_i - \gamma^{'}) + O(\rho^{-p\alpha}), i = 1, 2, ..., b_k,$$

where the summation is taken under the conditions $\gamma^{'} \in \Gamma(\rho^{\alpha})$ and $h_i - \gamma^{'} + t \in B_k(\gamma + t, p_1)$. It can be written in matrix form

$$(C - \Lambda_N I)(b(N, h_1), b(N, h_2), ...b(N, h_{b_k})) = O(\rho^{-p\alpha}),$$

where the right-hand side of this system is a vector having the norm $\parallel O(\rho^{-p\alpha}) \parallel = O(\sqrt{b_k}\rho^{-p\alpha})$. Now, taking into account that $\gamma + t \in \{h_i + t : i = 1, 2, ..., b_k\}$ and (7) holds, we have

$$c_4\rho^{-c\alpha} < (\sum_{i=1}^{b_k} \mid b(N, h_i) \mid^2)^{\frac{1}{2}} \leq \parallel (C - \Lambda_N I)^{-1} \parallel \sqrt{b_k}c_6\rho^{-p\alpha}, \tag{41}$$

$$\max_{i=1,2,...,b_k} \mid \Lambda_N - \lambda_i \mid^{-1} = \parallel (C - \Lambda_N I)^{-1} \parallel > c_4c_6^{-1}b_k^{-\frac{1}{2}}\rho^{-c\alpha+p\alpha}. \tag{42}$$

Since b_k is the number of the vectors of $B_k(\gamma+t, p_1)$, it follows from the definition of $B_k(\gamma + t, p_1)$ (see (26)) and the obvious relations $\mid B_k \mid = O(\rho^{\frac{k}{2}\alpha_{k+1}})$, $\mid \Gamma(p_1\rho^{\alpha}) \mid = O(\rho^{d\alpha})$ and $d\alpha < \frac{1}{2}\alpha_d$ (see the end of introduction), we get

$$b_k = O(\rho^{d\alpha+\frac{k}{2}\alpha_{k+1}}) = O(\rho^{\frac{d}{2}\alpha_d}) = O(\rho^{\frac{d}{2}3^d\alpha}), \forall k = 1, 2, ..., d - 1 \tag{43}$$

Thus formula (38) follows from (42) and (43).

(b) Let $\Lambda_N\ (t)$ be any eigenvalue of the operator $L_t(l, q(x))$ such that $\sqrt[2l]{\Lambda_N(t)} \in (\frac{3}{4}\rho, \frac{5}{4}\rho)$. Denote by D the set of all vectors $\gamma \in \Gamma$ satisfying (6). From (16), arguing as in the proof of Theorem 1(b), we obtain $\sum_{\gamma\in D} \mid b(N, \gamma) \mid^2 = 1 - O(\rho^{-2\alpha_1})$. Since $\mid D \mid = O(\rho^{d-1})$ (see the end of the introduction), there exists $\gamma \in D$ such that $\mid b(N, \gamma) \mid > c_7\rho^{-\frac{(d-1)}{2}} = c_7\rho^{-\frac{(d-1)m}{2}\alpha}$, that is, condition (7) for $c = \frac{(d-1)m}{2}$ holds. Now the proof of (b) follows from Theorem 1 (a) and Theorem 2(a), since either $\gamma + t \in U^1(\rho^{\alpha_1}, p)$ or $\gamma + t \in E_k^1 \backslash E_{k+1}^1$ for $k = 1, 2, ..., d-1$ (see (46)) ■

Remark 1 *Here we note that the non-resonance domain*
$U^l(c_8\rho^{\alpha_1},p)=(R(\frac{3}{2}\rho)\backslash R(\frac{1}{2}\rho))\backslash\bigcup_{\gamma_1\in\Gamma(p\rho^{\alpha})}V^l_{\gamma_1}(c_8\rho^{\alpha_1})$, *where*
$V^l_{\gamma_1}(c_8\rho^{\alpha_1})\equiv\{x:\mid\mid x\mid^{2l}-\mid x+\gamma_1\mid^{2l}\mid< c_8\rho^{\alpha_1}\}\cap(R(\frac{3}{2}\rho)\backslash R(\frac{1}{2}\rho))$, *has an asymptotically full measure on* $\mathbb{R}^d$ *in the sense that* $\frac{\mu(U^l\cap B(\rho))}{\mu(B(\rho))}$ *tends to* 1 *as* ρ *tends to infinity, where* $B(\rho)=\{x\in\mathbb{R}^d:\mid x\mid=\rho\}$. *By (30) it is enough to prove this for* $l=1$. *Clearly,* $B(\rho)\cap V^1_b(c_8\rho^{\alpha_1})$ *is the part of sphere* $B(\rho)$, *which is contained between two parallel hyperplanes*
$\{x:\mid x\mid^2-\mid x+b\mid^2=-c_8\rho^{\alpha_1}\}$ *and* $\{x:\mid x\mid^2-\mid x+b\mid^2=c_8\rho^{\alpha_1}\}$. *The distance of these hyperplanes from origin is* $O(\frac{\rho^{\alpha_1}}{|b|})$. *Therefore, the relations* $\mid\Gamma(p\rho^{\alpha})\mid=O(\rho^{d\alpha})$, *and* $\alpha_1+d\alpha<1-\alpha$ *(see the first inequality in (15)) imply*

$$\mu(B(\rho)\cap V^1_b(c_8\rho^{\alpha_1}))=O(\frac{\rho^{\alpha_1+d-2}}{\mid b\mid}),\ \mu(E^1_1\cap B(\rho))=O(\rho^{d-1-\alpha}),\tag{44}$$

$$\mu(U^1(c_8\rho^{\alpha_1},p)\cap B(\rho))=(1+O(\rho^{-\alpha}))\mu(B(\rho)).\tag{45}$$

If $x\in\cap^d_{i=1}V^1_{\gamma_i}(\rho^{\alpha_d})$, *then (35) holds for* $k=d$ *and* $n=1,2,...,d$. *Hence we have* $\mid x\mid=O(\rho^{\alpha_d+(d-1)\alpha})$. *It is impossible, since* $\alpha_d+(d-1)\alpha<1$ *(see the sixth inequality in (15)) and* $x\in B(\rho)$. *It means that* $(\cap^d_{i=1}V^1_{\gamma_i}(\rho^{\alpha_k}))\cap B(\rho)=\emptyset$ *for* $\rho\gg1$. *Thus for* $\rho\gg1$ *we have*

$$R(\frac{3}{2}\rho)\backslash R(\frac{1}{2}\rho)=(U^1(\rho^{\alpha_1},p)\cup(\cup^{d-1}_{s=1}(E^1_s\backslash E^1_{s+1}))).\tag{46}$$

Note that everywhere in this paper we use the big parameter ρ. *All considered eigenvalues* $|\gamma+t|^{2l}$ *of* $L_t(l,0)$ *satisfy the relations* $\frac{1}{2}\rho<|\gamma+t|<\frac{3}{2}\rho$. *Therefore in the asymptotic formulas instead of* $O(\rho^a)$ *one can take* $O(|\gamma+t|^a)$. *For simplicity, we often use* $O(\rho^a)$. *It is clear that the asymptotic formulas hold true if we replace* $U^l(\rho^{\alpha_1},p)$ *by* $U^l(c_8\rho^{\alpha_1},p)$, *where instead of* c_8 *one can write* $\frac{1}{2}$ *or* $\frac{3}{2}$. *Since* $V^l_b(\frac{1}{2}\rho^{\alpha_1})\subset(R(\frac{3}{2}\rho)\backslash R(\frac{1}{2}\rho))\cap W^l_{b,\alpha_1}(1)\subset V^l_b(\frac{3}{2}\rho^{\alpha_1})$, *in all considerations the resonance domain* $V^l_b(\rho^{\alpha_1})$ *can be replaced by* $W^l_{b,\alpha_1}(1)\cap(R(\frac{3}{2}\rho)\backslash R(\frac{1}{2}\rho))$.

Remark 2 *Here we note some properties of the known part*
$\mid\gamma+t\mid^{2l}+F_k(\gamma+t)$ *(see Theorem 1) of the non-resonance eigenvalues of* $L_t(l,q(x))$. *Denoting* $\gamma+t$ *by* x , *where* $\gamma+t\in U^1(\rho^{\alpha_1},p)$, *we prove*

$$\frac{\partial F_k(x)}{\partial x_i}=O(\rho^{2-2l-2\alpha_1+\alpha}),\forall i=1,2,...,d;\forall k=1,2,...\tag{47}$$

by induction on k. *Using (28) one can easily verify that if* $\mid x\mid\sim\rho$, *and*
$x\in U^1(\rho^{\alpha_1},p)$, *that is, if* $x\notin V^1_{\gamma_1}(\rho^{\alpha_1})$ *for* $\gamma_1\in\Gamma(p\rho^{\alpha})$, *then*

$$\mid x\mid^{2l}-\mid x-\gamma_1\mid^{2l}\sim\rho^{2l-2}(\mid x\mid^2-\mid x-\gamma_1\mid^2),$$
$$\mid x\mid^{2l-2}-\mid x-\gamma_1\mid^{2l-2}\sim\rho^{2l-4}(\mid x\mid^2-\mid x-\gamma_1\mid^2),$$

$$\mid\mid x\mid^2-\mid x-\gamma_1\mid^2\mid>\rho^{\alpha_1},$$

$$\frac{\partial}{\partial x_i}(\frac{1}{\mid x \mid^{2l} - \mid x-\gamma_1 \mid^{2l}}) =$$

$$-\frac{2lx_i(\mid x \mid^{2l-2} - \mid x-\gamma_1 \mid^{2l-2})}{(\mid x \mid^{2l} - \mid x-\gamma_1 \mid^{2l})^2})+ \tag{48}$$

$$\frac{2\gamma_1(i) \mid x-\gamma_1 \mid^{2l-2}}{(\mid x \mid^{2l} - \mid x-\gamma_1 \mid^{2l})^2} = O(\rho^{2-2l-2\alpha_1+\alpha}),$$

where $(\gamma_1(1), \gamma_1(2), ..., \gamma_1(d)) = \gamma_1 \in \Gamma(p\rho^{\alpha})$ *and hence* $\gamma_1(i) = O(\rho^{\alpha})$. *Now (47) for* $k=1$ *follows from (4) and (48). Suppose that (47) holds for* $k=s$. *Using this and (24), replacing* $\mid x \mid^{2l}$ *by* $\mid x \mid^{2l} + F_s(x)$ *in (48), and evaluating as above, we obtain*

$$\frac{\partial}{\partial x_i}(\frac{1}{\mid x \mid^{2l} + F_s(x) - \mid x-\gamma_1 \mid^{2l}}) =$$

$$O(\rho^{2l-2-2\alpha_1+\alpha}) - \frac{\frac{\partial F_s(x)}{\partial x_i}}{(\mid x \mid^{2l} + F_s - \mid x-\gamma_1 \mid^{2l})^2} =$$

$$O(\rho^{2-2l-2\alpha_1+\alpha}) + O(\rho^{2-2l-4\alpha_1+\alpha}) = O(\rho^{2-2l-2\alpha_1+\alpha})$$

This formula with the definition of F_k *implies (47) for* $k=s+1$.

3 Asymptotic Formulas for Bloch Functions

In this section using the asymptotic formulas for the eigenvalues and the simplicity conditions (12), (13), we prove the asymptotic formulas for the Bloch functions with a quasimomentum of the simple set B.

Theorem 3 *If* $\gamma + t \in B$, *then there exists a unique eigenvalue* $\Lambda_N(t)$ *satisfying (5) for* $k = 1, 2, ..., [\frac{p}{3}]$, *where* p *is defined in (3). This is a simple eigenvalue and the corresponding eigenfunction* $\Psi_{N,t}(x)$ *of* $L(l, q(x))$ *satisfies (10) if* $q(x) \in W_2^{s_0}(F)$, *where* $s_0 = \frac{3d-1}{2}(3^d + d + 2) + \frac{1}{4}d3^d + d + 6$.

Proof. By Theorem 1(b) if $\gamma + t \in B \subset U^1(\rho^{\alpha_1}, p)$, then there exists an eigenvalue $\Lambda_N(t)$ satisfying (5) for $k = 1, 2, ..., [\frac{1}{3}(p - \frac{1}{2}m(d-1))]$. Since $k_1 = [\frac{d}{3\alpha}] + 2 \le \frac{1}{3}(p - \frac{1}{2}m(d-1))$ (see the third inequality in (15)) formula (5) holds for $k = k_1$. Therefore using (5), the relation $3k_1\alpha > d + 2\alpha$ (see the fifth inequality in (15)), and notations $F(\gamma + t) = \mid \gamma + t \mid^{2l} + F_{k_1-1}(\gamma + t)$, $\varepsilon_1 = \rho^{-d-2\alpha}$ (see Step 1 in introduction), we obtain

$$\Lambda_N(t) = F(\gamma + t) + o(\varepsilon_1). \tag{49}$$

Let Ψ_N be any normalized eigenfunction corresponding to Λ_N. Since the normalized eigenfunction is defined up to constant of modulus 1, without loss of generality it can assumed that $\arg b(N, \gamma) = 0$, where $b(N, \gamma) = (\Psi_N, e^{i(\gamma+t,x)})$.

Therefore to prove (10) it suffices to show that (14) holds. To prove (14) first we estimate $\sum_{\gamma^{'}\notin K}\mid b(N,\gamma^{'})\mid^{2}$and then $\sum_{\gamma^{'}\in K\backslash\{\gamma\}}\mid b(N,\gamma^{'})\mid^{2}$, where K is defined in (12), (13). Using (102), the definition of K, and (16), we get

$$\mid\Lambda_{N}-\mid\gamma^{'}+t\mid^{2l}\mid>\frac{1}{4}\rho^{\alpha_{1}},\ \forall\gamma^{'}\notin K,\tag{50}$$

$$\sum_{\gamma^{'}\notin K}\mid b(N,\gamma^{'})\mid^{2}=\parallel q(x)\Psi_{N}\parallel^{2}O(\rho^{-2\alpha_{1}})=O(\rho^{-2\alpha_{1}}).$$

If $\gamma^{'}\in K$, then by (49) and by definition of K, it follows that

$$\mid\Lambda_{N}-\mid\gamma^{'}+t\mid^{2l}\mid<\frac{1}{2}\rho^{\alpha_{1}}\tag{51}$$

Now we prove that the simplicity conditions (12), (13) imply

$$\mid b(N,\gamma^{'})\mid\leq c_{4}\rho^{-c\alpha},\ \forall\gamma^{'}\in K\backslash\{\gamma\},\tag{52}$$

where $c=p-dm-\frac{1}{4}d3^{d}-3$. The conditions $\gamma^{'}\in K$, $\gamma+t\in B$ and (24) imply the inclusion $\gamma^{'}+t\in R(\frac{3}{2}\rho)\backslash R(\frac{1}{2}\rho)$. If for $\gamma^{'}+t\in U^{1}(\rho^{\alpha_{1}},p)$ and $\gamma^{'}\in K\backslash\{\gamma\}$ the inequality in (52) is not true, then by (51) and Theorem 1(a), we have

$$\Lambda_{N}=\mid\gamma^{'}+t\mid^{2l}+F_{k-1}(\gamma^{'}+t)+O(\rho^{-3k\alpha})\tag{53}$$

for $k=1,2,...,[\frac{1}{3}(p-c)]=[\frac{1}{3}(dm+\frac{1}{4}d3^{d}+3)]$. Since $\alpha=\frac{1}{m}$ and $k_{1}\equiv[\frac{d}{3\alpha}]+2<\frac{1}{3}(dm+\frac{1}{4}d3^{d}+3)$, the formula (53) holds for $k=k_{1}$. Therefore arguing as in the prove of (49), we get $\Lambda_{N}-F(\gamma^{'}+t)=o(\varepsilon_{1})$. This with (49) contradicts (12). Similarly, if the inequality in (52) does not hold for $\gamma^{'}+t\in(E_{k}^{1}\backslash E_{k+1}^{1})$ and $\gamma^{'}\in K$, then by Theorem 2(a)

$$\Lambda_{N}=\lambda_{j}(\gamma^{'}+t)+O(\rho^{-(p-c-\frac{1}{4}d3^{d})\alpha}),\tag{54}$$

where $(p-c-\frac{1}{4}d3^{d})\alpha=(dm+3)\alpha>d+2\alpha$. Hence we have $\Lambda_{N}-\lambda_{j}(\gamma^{'}+t)=o(\varepsilon_{1})$. This with (49) contradicts (13). So the inequality in (52) holds. Therefore, using $\mid K\mid=O(\rho^{d-1})$, $m\alpha=1$, we get

$$\sum_{\gamma^{'}\in K\backslash\{\gamma\}}\mid b(N,\gamma^{'})\mid^{2}=O(\rho^{-(2c-q(d-1))\alpha})=O(\rho^{-(2p-(3d-1)q-\frac{1}{2}d3^{d}-6)\alpha}).\tag{55}$$

If $s=s_{0}$, that is, $p=s_{0}-d$, then $2p-(3d-1)m-\frac{1}{2}d3^{d}-6=6$. Since $\alpha_{1}=3\alpha$, the equality (55) and the equality in (50) imply (14). Thus we proved that the equality (10) holds for any normalized eigenfunction Ψ_{N} corresponding to any eigenvalue Λ_{N} satisfying (5). If there exist two different eigenvalues or multiple eigenvalue satisfying (5), then there exist two orthogonal normalized eigenfunction satisfying (10), which is impossible. Therefore Λ_{N} is a simple eigenvalue. It follows from Theorem 1(a) that Λ_{N} satisfies (5) for $k=1,2,...,[\frac{p}{3}]$, since the inequality (7) holds for $c=0$ (see (10)). ■

Remark 3 *Since for $\gamma + t \in B$ there exists a unique eigenvalue satisfying (5), (49) we denote this eigenvalue by $\Lambda(\gamma+t)$. Since this eigenvalue is simple, we denote the corresponding eigenfunction by $\Psi_{\gamma+t}(x)$. By Theorem 3 this eigenfunction satisfies (10). Clearly, for $\gamma+t \in B$ there exists a unique index $N \equiv N(\gamma+t)$ such that $\Lambda(\gamma+t) = \Lambda_{N(\gamma+t)})$ and $\Psi_{\gamma+t}(x) = \Psi_{N(\gamma+t)}(x))$.*

Now we prove the asymptotic formulas of arbitrary order for $\Psi_{\gamma+t}(x)$.

Theorem 4 *If $\gamma+t \in B$, then the eigenfunction $\Psi_{\gamma+t}(x) \equiv \Psi_{N(\gamma+t)}(x)$ corresponding to the eigenvalue $\Lambda_N \equiv \Lambda(\gamma+t)$ satisfies formulas (11), for*
$k = 1, 2, ..., n$, where $n = [\frac{1}{6}(2p - (3d-1)m - \frac{1}{2}d3^d - 6)]$,

$$\Phi_0(x) = 0,\ \Phi_1(x) = \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} \frac{q_{\gamma_1} e^{i(\gamma+t+\gamma_1,x)}}{(\mid \gamma+t \mid^{2l} - \mid \gamma+\gamma_1+t \mid^{2l})},$$

and $\Phi_{k-1}(x)$ for $k > 2$ is a linear combination of $e^{i(\gamma+t+\gamma^{'},x)}$ for $\gamma^{'} \in \Gamma((k-1)\rho^\alpha) \cup \{0\}$ with coefficients (61), (62).

Proof. By Theorem 3, formula (11) for $k=1$ is proved. To prove formula (11) for arbitrary $k \leq n$ we prove the following equivalent relations

$$\sum_{\gamma^{'} \in \Gamma^c(k-1)} \mid b(N, \gamma+\gamma^{'}) \mid^2 = O(\rho^{-2k\alpha_1}), \tag{56}$$

$$\Psi_N = b(N,\gamma)e^{i(\gamma+t,x)} + \sum_{\gamma^{'} \in \Gamma((k-1)\rho^\alpha)} b(N,\gamma+\gamma^{'})e^{i(\gamma+t+\gamma^{'},x)} + H_k(x), \tag{57}$$

where $\Gamma^c(j) \equiv \Gamma \backslash (\Gamma(j\rho^\alpha) \cup \{0\})$ and $\parallel H_k(x) \parallel = O(\rho^{-k\alpha_1})$. The case $k=1$ is proved due to (14). Assume that (56) is true for $k=j$. Then using (57) for $k=j$, and (3), we have $\Psi_N(x)(q(x)) = H(x) + O(\rho^{-j\alpha_1})$, where $H(x)$ is a linear combination of $e^{i(\gamma+t+\gamma^{'},x)}$ for $\gamma^{'} \in \Gamma(j\rho^\alpha) \cup \{0\}$. Hence $(H(x), e^{i(\gamma+t+\gamma^{'},x)}) = 0$ for $\gamma^{'} \in \Gamma^c(j)$. So using (16) and (50), we get

$$\sum_{\gamma^{'}} \mid b(N,\gamma+\gamma^{'}) \mid^2 = \sum_{\gamma^{'}} \mid \frac{(O(\rho^{-j\alpha_1}), e^{i(\gamma+t+\gamma^{'},x)})}{\Lambda_N - \mid \gamma+\gamma^{'}+t \mid^{2l}} \mid^2 = O(\rho^{-2(j+1)\alpha_1}), \tag{58}$$

where the summation is taken under conditions $\gamma^{'} \in \Gamma^c(j)$, $\gamma+\gamma^{'} \notin K$. On the other hand, using $\alpha_1 = 3\alpha$, (108), and the definition of n, we obtain

$$\sum_{\gamma^{'} \in K \backslash \{\gamma\}} \mid b(N,\gamma^{'}) \mid^2 = O(\rho^{-2n\alpha_1}).$$

This with (58) implies (56) for $k = j+1$. Thus (57) is also proved. Here $b(N,\gamma)$ and $b(N,\gamma+\gamma^{'})$ for $\gamma^{'} \in \Gamma((n-1)\rho^\alpha)$ can be calculated as follows. First we express $b(N,\gamma+\gamma^{'})$ by $b(N,\gamma)$. For this we apply (18) for $b(N,\gamma+\gamma^{'})$, where $\gamma^{'} \in \Gamma((n-1)\rho^\alpha)$, that is, in (18) replace $\gamma^{'}$ by $\gamma+\gamma^{'}$. Iterate it n times and

every times isolate the terms with multiplicand $b(N,\gamma)$. In other word apply (18) for $b(N,\gamma+\gamma^{'})$ and isolate the terms with multiplicand $b(N,\gamma)$. Then apply (18) for $b(N,\gamma+\gamma^{'}-\gamma_1)$ when $\gamma^{'}-\gamma_1\neq 0$. Then apply (18) for $b(N,\gamma+\gamma^{'}-\sum_{i=1}^{2}\gamma_i)$ when $\gamma^{'}-\sum_{i=1}^{2}\gamma_i\neq 0$, etc. Apply (18) for $b(N,\gamma+\gamma^{'}-\sum_{i=1}^{j}\gamma_i)$ when $\gamma^{'}-\sum_{i=1}^{j}\gamma_i\neq 0$, where $\gamma_i\in\Gamma(\rho^{\alpha})$, $j=3,4,...,n-1$. Then using (4) and the relations $\mid\Lambda_N-\mid\gamma+t+\gamma^{'}-\sum_{i=1}^{j}\gamma_i\mid^{2l}\mid>\frac{1}{2}\rho^{\alpha_1}$ (see (20) and take into account that $\gamma^{'}-\sum_{i=1}^{j}\gamma_i\in\Gamma(p\rho^{\alpha})$, since $p>2n$), $\Lambda_N=P(\gamma+t)+O(\rho^{-n\alpha_1})$, where $P(\gamma+t)=\mid\gamma+t\mid^{2l}+F_{[\frac{p}{3}]}(\gamma+t)$ (see Theorem 3), we obtain

$$b(N,\gamma+\gamma^{'})=\sum_{k=1}^{n-1}A_k(\gamma^{'})b(N,\gamma)+O(\rho^{-n\alpha_1}),\qquad (59)$$

where

$$A_1(\gamma^{'})=\frac{q_{\gamma^{'}}}{P(\gamma+t)-\mid\gamma+\gamma^{'}+t\mid^{2l}}=\frac{q_{\gamma^{'}}}{\mid\gamma+t\mid^{2l}-\mid\gamma+\gamma^{'}+t\mid^{2l}}+O(\tfrac{1}{\rho^{3\alpha_1}}),$$

$$A_k(\gamma^{'})=\sum_{\gamma_1,...,\gamma_{k-1}}\frac{q_{\gamma_1}q_{\gamma_2}...q_{\gamma_{k-1}}q_{\gamma^{'}-\gamma_1-\gamma_2-...-\gamma_{k-1}}}{\prod_{j=0}^{k-1}(P(\gamma+t)-\mid\gamma+t+\gamma^{'}-\sum_{i=1}^{j}\gamma_i\mid^{2l})}=O(\rho^{-k\alpha_1}),$$

$$\sum_{\gamma^*\in\Gamma((n-1)\rho^{\alpha})}\mid A_1(\gamma^*)\mid^2=O(\rho^{-2\alpha_1}),\quad\sum_{\gamma^*\in\Gamma((n-1)\rho^{\alpha})}\mid A_k(\gamma^*)\mid=O(\rho^{-k\alpha_1})\qquad (60)$$

for $k>1$. Now from (57) for $k=n$ and (59), we obtain

$$\Psi_N(x)=b(N,\gamma)e^{i(\gamma+t,x)}+$$
$$\sum_{\gamma^*\in\Gamma((n-1)\rho^{\alpha})}\sum_{k=1}^{n-1}(A_k(\gamma^*)b(N,\gamma)+O(\rho^{-n\alpha_1}))e^{i(\gamma+t+\gamma^*,x)})+H_n(x).$$

Using the equalities $\parallel\Psi_N\parallel=1$, $\arg b(N,\gamma)=0$, $\parallel H_n\parallel=O(\rho^{-n\alpha_1})$ and taking into account that the functions $e^{i(\gamma+t,x)}$, $H_n(x)$, $e^{i(\gamma+t+\gamma^*,x)}$, $(\gamma^*\in\Gamma((n-1)\rho^{\alpha}))$ are orthogonal, we get

$$1=\mid b(N,\gamma)\mid^2+\sum_{k=1}^{n-1}(\sum_{\gamma^*\in\Gamma((n-1)\rho^{\alpha})}\mid A_k(\gamma^*)b(N,\gamma)\mid^2+O(\rho^{-n\alpha_1})),$$

$$b(N,\gamma)=(1+\sum_{k=1}^{n-1}(\sum_{\gamma^*\in\Gamma((n-1)\rho^{\alpha})}\mid A_k(\gamma^*)\mid^2))^{-\frac{1}{2}}+O(\rho^{-n\alpha_1}))\qquad (61)$$

(see the second equality in (60)). Thus from (59), we obtain

$$b(N,\gamma+\gamma^{'})=(\sum_{k=1}^{n-1}A_k(\gamma^{'}))(1+\sum_{k=1}^{n-1}\sum_{\gamma^*}\mid A_k(\gamma^*)\mid^2)^{-\frac{1}{2}}+O(\rho^{-n\alpha_1}).\qquad (62)$$

Consider the case $n=2$. By (61), (60), (62) we have $b(N,\gamma)=1+O(\rho^{-2\alpha_1})$,

$$b(N,\gamma+\gamma^{'})=A_1(\gamma^{'})+O(\rho^{-2\alpha_1})=\frac{q_{\gamma^{'}}}{\mid\gamma+t\mid^{2l}-\mid\gamma+\gamma^{'}+t\mid^{2l}}+O(\rho^{-2\alpha_1})$$

for all $\gamma^{'}\in\Gamma(\rho^{\alpha})$. These and (57) for $k=2$ imply the formula for Φ_1 ■

4 Simple Sets and Bethe-Sommerfeld conjecture

In this section we construct a part of the simple set B in the neighbourhood of the surface $S_\rho \equiv \{x \in U^1(2\rho^{\alpha_1}, p) : F(x) = \rho^{2l}\}$, where $F(x) =| x |^{2l} + F_{k_1-1}(x)$ for $x = \gamma + t$ is defined in (49) and in introduction (see step 1). Due to (49) it is natural to call S_ρ the approximated isoenergetic surfaces in the non-resonance domain. As we noted in introduction (see the inequality (12)) the non-resonance eigenvalue $\Lambda(\gamma + t)$, where $\Lambda(\gamma + t) = \Lambda_{N(\gamma+t)}(t)$ is defined in Remark 3, does not coincide with other non-resonance eigenvalue $\Lambda(\gamma + t + b)$ if $| F(\gamma + t) - F(\gamma + t + b) |> 2\varepsilon_1$ for $\gamma + t + b \in U^1(\rho^{\alpha_1}, p)$ and $b \in \Gamma\backslash\{0\}$. Therefore we eliminate

$$P_b \equiv \{x : x, x + b \in U^1(\rho^{\alpha_1}, p), | F(x) - F(x + b) |< 3\varepsilon_1\} \tag{63}$$

for $b \in \Gamma\backslash\{0\}$ from S_ρ. Denote the remaining part of S_ρ by $S_\rho^{'}$. Then we consider the ε neighbourhood

$U_\varepsilon(S_\rho^{'}) = \cup_{a\in S_\rho^{'}} U_\varepsilon(a)\}$ of $S_\rho^{'}$, where $U_\varepsilon(a) = \{x \in \mathbb{R}^d :| x - a |< \varepsilon\}$,

and prove that in this set the first simplicity condition (12) holds (see Lemma 2(a)). Denote by $Tr(E) \equiv \{\gamma + x \in U_\varepsilon(S_\rho^{'}) : \gamma \in \Gamma, x \in E\}$ and

$Tr_{F^\star}(E) \equiv \{\gamma + x \in F^\star : \gamma \in \Gamma, x \in E\}$ the translations of $E \subset \mathbb{R}^d$ into $U_\varepsilon(S_\rho^{'})$ and $F^\star$ respectively. In order that the second simplicity condition (13) holds, we discard from $U_\varepsilon(S_\rho^{'})$ the translation $Tr(A(\rho))$ of

$$A(\rho) \equiv \cup_{k=1}^{d-1}(\cup_{\gamma_1,\gamma_2,\ldots,\gamma_k\in\Gamma(p\rho^\alpha)}(\cup_{i=1}^{b_k} A_{k,i}(\gamma_1, \gamma_2, ..., \gamma_k))), \tag{64}$$

where $A_{k,i}(\gamma_1, ..., \gamma_k) =$
$\{x \in (\cap_{i=1}^{k} V_{\gamma_i}^1(\rho^{\alpha_k})\backslash E_{k+1}^1) \cap K_\rho : \lambda_i(x) \in (\rho^{2l} - 3\varepsilon_1, \rho^{2l} + 3\varepsilon_1)\}$,
$\lambda_i(x)$, b_k is defined in Theorem 2 and

$$K_\rho = \{x \in \mathbb{R}^d :|| x |^{2l} - \rho^{2l} |< \rho^{\alpha_1}\}. \tag{65}$$

As a result, we construct the part $U_\varepsilon(S_\rho^{'})\backslash Tr(A(\rho))$ of the simple set B (see Theorem 5(a)), which contains the intervals $\{a + sb : s \in [-1, 1]\}$ such that $\Lambda(a - b) < \rho^{2l}$, $\Lambda(a + b) > \rho^{2l}$ and $\Lambda(\gamma + t)$ is continuous on this intervals. Hence there exists $\gamma + t$ such that $\Lambda(\gamma + t) = \rho^{2l}$. It implies the validity of Bethe-Sommerfeld conjecture for $L(l, q)$.

Lemma 2 *(a) If $x \in U_\varepsilon(S_\rho^{'})$ and $x + b \in U^1(\rho^{\alpha_1}, p)$, where $b \in \Gamma$, then*

$$| F(x) - F(x + b) |> 2\varepsilon_1, \tag{66}$$

where $\varepsilon = \frac{\varepsilon_1}{7l\rho^{2l-1}}, \varepsilon_1 = \rho^{-d-2\alpha}$, hence for $\gamma + t \in U_\varepsilon(S_\rho^{'})$ the first simplicity condition (12) holds.

(b) If $x \in U_\varepsilon(S_\rho^{'})$, then $x + b \notin U_\varepsilon(S_\rho^{'})$ for all $b \in \Gamma$.

(c) If $E \subset \mathbb{R}^d$ is bounded set, then $\mu(Tr(E)) \leq \mu(E)$.

(d) If $E \subset U_\varepsilon(S_\rho^{'})$, then $\mu(Tr_{F^\star}(E)) = \mu(E)$.

Proof. (a) If $x \in U_\varepsilon(S_\rho^{'})$, then there exists a point a in $S_\rho^{'}$ such that $x \in U_\varepsilon(a)$. Since $S_\rho^{'} \cap P_b = \emptyset$ (see (63) and def. of $S_\rho^{'}$), we have

$$| F(a) - F(a+b) | \geq 3\varepsilon_1 \tag{67}$$

On the other hand, using (47) and the obvious relations
$| x | < \rho + 1$, $| x - a | < \varepsilon$, $| x + b - a - b | < \varepsilon$, we obtain

$$| F(x) - F(a) | < 3l\rho^{2l-1}\varepsilon, | F(x+b) - F(a+b) | < 3l\rho^{2l-1}\varepsilon \tag{68}$$

These inequalities together with (67) give (66), since $6l\rho^{2l-1}\varepsilon < \varepsilon_1$.

(b) If x and $x+b$ lie in $U_\varepsilon(S_\rho^{'})$, then there exist points a and c in $S_\rho^{'}$ such that $x \in U_\varepsilon(a)$ and $x + b \in U_\varepsilon(c)$. Repeating the proof of (68), we get
$| F(c) - F(x+b) | < 3l\rho^{2l-1}\varepsilon$. This, the first inequality in (68), and the relations $F(a) = \rho^{2l}, F(c) = \rho^{2l}$ (see the definition of S_ρ) give
$| F(x) - F(x+b) | < \varepsilon_1$, which contradicts (66).

(c) Clearly, for any bounded set E there are only finite number of the vectors $\gamma_1, \gamma_2, ..., \gamma_s$ such that $E(k) \equiv (E + \gamma_k) \cap U_\varepsilon(S_\rho^{'}) \neq \emptyset$ for $k = 1, 2, ..., s$ and $Tr(E)$ is the union of the sets $E(k)$. For $E(k) - \gamma_k$ we have the relations $\mu(E(k) - \gamma_k) = \mu(E(k))$, $E(k) - \gamma_k \subset E$. Moreover, by (b)
$(E(k) - \gamma_k) \cap (E(j) - \gamma_j) = \emptyset$ for $k \neq j$. Therefore (c) is true.

(d) Now let $E \subset U_\varepsilon(S_\rho^{'})$. Then by (b) the set E can be divided into finite number of the pairwise disjoint sets $E_1, E_2, ..., E_n$ such that there exist the vectors $\gamma_1, \gamma_2, ..., \gamma_n$ satisfying $(E_k + \gamma_k) \subset F^\star$, $(E_k + \gamma_k) \cap (E_j + \gamma_j) \neq \emptyset$ for $k, j = 1, 2, ..., n$ and $k \neq j$. Using $\mu(E_k + \gamma_k) = \mu(E_k)$, we get the proof of (d), because $Tr_{F^\star}(E)$ and E are union of the pairwise disjoint sets $E_k + \gamma_k$ and E_k for $k = 1, 2, ..., n$ respectively ■

Theorem 5 (a) *The set* $U_\varepsilon(S_\rho^{'}) \backslash Tr(A(\rho))$ *is a subset of* B, *hence if* $\gamma + t$ *lies in this subset, then Theorem 3 and Theorem 4 hold. For every connected open subset* E *of* $U_\varepsilon(S_\rho^{'}) \backslash Tr(A(\rho)$ *there exists a unique index* N *such that*
$\Lambda(\gamma + t) = \Lambda_N(t)$ *for* $\gamma + t \in E$, *where* $\Lambda(\gamma + t)$ *is defined in Remark 3.*

(b) *For the part* $V_\rho \equiv S_\rho^{'} \backslash U_\varepsilon(Tr(A(\rho)))$ *of the approximated isoenergetic surface* S_ρ *the following holds*

$$\mu(V_\rho) > (1 - c_9\rho^{-\alpha}))\mu(B(\rho)). \tag{69}$$

Moreover, $U_\varepsilon(V_\rho)$ *lies in the subset* $U_\varepsilon(S_\rho^{'}) \backslash Tr(A(\rho))$ *of the simple set* B.

(c) *The number* ρ^{2l} *for* $\rho \gg 1$ *lies in the spectrum of* $L(l, q(x))$, *that is, the number of the gaps in the spectrum of* $L(l, q(x))$ *is finite, where* $l \geq 1$, $q(x) \in W_2^{s_0}(\mathbb{R}^d/\Omega)$, $d \geq 2$, $s_0 = \frac{3d-1}{2}(3^d + d + 2) + \frac{1}{4}d3^d + d + 6$, *and* Ω *is an arbitrary lattice.*

Proof. (a) To prove that $U_\varepsilon(S_\rho^{'}) \backslash Tr(A(\rho)) \subset B$ we need to show that for each point $\gamma + t$ of $U_\varepsilon(S_\rho^{'}) \backslash Tr(A(\rho))$ the simplicity conditions (12), (13) hold and $U_\varepsilon(S_\rho^{'}) \backslash Tr(A(\rho)) \subset U^1(\rho^{\alpha_1}, p)$. By lemma 2(a), the condition (12) holds. Now we prove that (13) holds too. Since $\gamma + t \in U_\varepsilon(S_\rho^{'})$, there exists $a \in S_\rho^{'}$

such that $\gamma+t\in U_\varepsilon(a)$. The first inequality in (68) and equality $F(a)=\rho^{2l}$ imply

$$F(\gamma+t)\in(\rho^{2l}-\varepsilon_1,\rho^{2l}+\varepsilon_1) \tag{70}$$

for $\gamma+t\in U_\varepsilon(S_\rho^{'})$. On the other hand $\gamma+t\notin Tr(A(\rho))$. It means that for any $\gamma^{'}\in\Gamma$, we have $\gamma^{'}+t\notin A(\rho)$. If $\gamma^{'}\in K$ and $\gamma^{'}+t\in E_k^1\backslash E_{k+1}^1$, then by definition of K (see introduction) the inequality $\mid F(\gamma+t)-\mid\gamma^{'}+t\mid^{2l}\mid<\frac{1}{3}\rho^{\alpha_1}$ holds. This and (70) imply that $\gamma^{'}+t\in(E_k^1\backslash E_{k+1}^1)\cap K_\rho$ (see (65) for the definition of K_ρ). Since $\gamma^{'}+t\notin A(\rho)$, we have $\lambda_i(\gamma^{'}+t)\notin(\rho^{2l}-3\varepsilon_1,\rho^{2l}+3\varepsilon_1)$ for $\gamma^{'}\in K$ and $\gamma^{'}+t\in E_k^1\backslash E_{k+1}^1$. Therefore (13) follows from (70). Moreover, it is clear that the inclusion $S_\rho^{'}\subset U^1(2\rho^{\alpha_1},p)$ (see definition of S_ρ and $S_\rho^{'}$) implies that $U_\varepsilon(S_\rho^{'})\subset U^1(\rho^{\alpha_1},p)$. Thus $U_\varepsilon(S_\rho^{'})\backslash Tr(A(\rho))\subset B$.

Now let E be a connected open subset of $U_\varepsilon(S_\rho^{'})\backslash Tr(A(\rho)\subset B$. By Theorem 3 and Remark 3 for $a\in E\subset U_\varepsilon(S_\rho^{'})\backslash Tr(A(\rho)$ there exists a unique index $N(a)$ such that $\Lambda(a)=\Lambda_{N(a)}(a)$, $\Psi_a(x)=\Psi_{N(a),a}(x)$, $\mid(\Psi_{N(a),a}(x),e^{i(a,x)})\mid^2>\frac{1}{2}$ and $\Lambda(a)$ is a simple eigenvalue. On the other hand, for fixed N the functions $\Lambda_N(t)$ and $(\Psi_{N,t}(x),e^{i(t,x)})$ are continuous in a neighborhood of a if $\Lambda_N(a)$ is a simple eigenvalue. Therefore for each $a\in E$ there exists a neighborhood $U(a)\subset E$ of a such that $\mid(\Psi_{N(a),y}(x),e^{i(y,x)})\mid^2>\frac{1}{2}$, for $y\in U(a)$. Since for $y\in E$ there is a unique integer $N(y)$ satisfying $\mid(\Psi_{N(y),y}(x),e^{i(y,x)})\mid^2>\frac{1}{2}$, we have $N(y)=N(a)$ for $y\in U(a)$. Hence we proved that

$$\forall a\in E,\exists U(a)\subset E: N(y)=N(a),\forall y\in U(a). \tag{71}$$

Now let a_1 and a_2 be two points of E , and let $C\subset E$ be the arc that joins these points. Let $U(y_1),U(y_2),...,U(y_k)$ be a finite subcover of the open cover $\cup_{a\in C}U(a)$ of the compact C, where $U(a)$ is the neighborhood of a satisfying (71). By (71), we have $N(y)=N(y_i)=N_i$ for $y\in U(y_i)$. Clearly, if

$U(y_i)\cap U(y_j)\neq\emptyset$, then $N_i=N(z)=N_j$, where $z\in U(y_i)\cap U(y_j)$. Thus $N_1=N_2=...=N_k$ and $N(a_1)=N(a_2)$.

(b) To prove the inclusion $U_\varepsilon(V_\rho)\subset U_\varepsilon(S_\rho^{'})\backslash Tr(A(\rho))$ we need to show that if $a\in V_\rho$, then $U_\varepsilon(a)\subset U_\varepsilon(S_\rho^{'})\backslash Tr(A(\rho))$. This is clear, since the relations $a\in V_\rho\subset S_\rho^{'}$ imply that $U_\varepsilon(a)\subset U_\varepsilon(S_\rho^{'})$ and the relation $a\notin U_\varepsilon(Tr(A(\rho)))$ implies that $U_\varepsilon(a)\cap Tr(A(\rho))=\emptyset$. To prove (69) first we estimate the measure of $S_\rho,S_\rho^{'},U_{2\varepsilon}(A(\rho))$, namely we prove

$$\mu(S_\rho)>(1-c_{10}\rho^{-\alpha})\mu(B(\rho)), \tag{72}$$

$$\mu(S_\rho^{'})>(1-c_{11}\rho^{-\alpha})\mu(B(\rho)), \tag{73}$$

$$\mu(U_{2\varepsilon}(A(\rho)))=O(\rho^{-\alpha})\mu(B(\rho))\varepsilon \tag{74}$$

(see below, Estimations 1, 2, 3). The estimation (69) of the measure of the set V_ρ is done in Estimation 4 by using Estimations 1, 2, 3.

(c) Since $F(a)=\rho^{2l}$ for $a=(a_1,a_2,...,a_d)=\sum_{i=1}^d a_ie_i\in V_\rho\subset S_\rho$, it follows from (24) that $\rho-1<\mid a\mid<\rho+1$, and there exists an index i such that $\mid a_i\mid>\frac{1}{d}\rho$.

Without loss of generality it can be assumed that $a_i > 0$. Then (47) and (49) imply that $F(a-\varepsilon e_i) < \rho^{2l}-c_{11}\varepsilon_1$, $F(a+\varepsilon e_i) > \rho^{2l}+c_{11}\varepsilon_1$ and $\Lambda(a-\varepsilon e_i) < \rho^{2l}$, $\Lambda(a+\varepsilon e_i) > \rho^{2l}$. Since $\Lambda(\gamma+t)$ is continuous in $U_\varepsilon(a) \subset U_\varepsilon(S_\rho^{'})\backslash Tr(A(\rho)$ (see Theorem 5(a)), there exists $y(a,i) \in (a-\varepsilon e_i, a+\varepsilon e_i)$ such that $\Lambda(y(a,i)) = \rho^{2l}$. The Theorem is proved ■

In Estimations 1-4 we use the notations: $G(+i,a) = \{x \in G, x_i > a\}$, $G(-i,a) = \{x \in G, x_i < -a\}$, where $x = (x_1, x_2, ..., x_d), a > 0$. It is not hard to verify that for any subset G of $U_\varepsilon(S_\rho^{'}) \cup U_{2\varepsilon}(A(\rho))$, that is, for all considered sets G in these estimations, and for any $x \in G$ the followings hold

$$\rho - 1 <\mid x \mid< \rho + 1,\ G \subset (\cup_{i=1}^{d}(G(+i, \rho d^{-1}) \cup G(-i, \rho d^{-1})) \tag{75}$$

Indeed, if $x \in S_\rho^{'}$, then $F(x) = \rho^{2l}$ and by (24) we have $\mid x \mid= \rho + O(\rho^{-1-\alpha_1})$. Hence the inequalities in (75) hold for $x \in U_\varepsilon(S_\rho^{'})$. If $x \in A(\rho)$, then by definition of $A(\rho)$, we have $x \in K_\rho$, and hence $\mid x \mid= \rho + O(\rho^{-1+\alpha_1})$. Therefore the inequalities in (75) hold for $x \in U_{2\varepsilon}(A(\rho))$ too. The inclusion in (75) follows from these inequalities.

If $G \subset S_\rho$, then by (47) we have $\frac{\partial F(x)}{\partial x_k} > 0$ for $x \in G(+k, \rho^{-\alpha})$. Therefore to calculate the measure of $G(+k,a)$ for $a \geq \rho^{-\alpha}$ we use the formula

$$\mu(G(+k,a)) = \int\limits_{\Pr_k(G(+k,a))} (\frac{\partial F}{\partial x_k})^{-1} \mid grad(F) \mid dx_1...dx_{k-1}dx_{k+1}...dx_d, \tag{76}$$

where $\Pr_k(G) \equiv \{(x_1, x_2, ..., x_{k-1}, x_{k+1}, x_{k+2}, ..., x_d) : x \in G\}$ is the projection of G on the hyperplane $x_k = 0$. Instead of $\Pr_k(G)$ we write $\Pr(G)$ if k is unambiguous. If D is $s-$dimensional subset of $\mathbb{R}^s$, then to estimate $\mu(D)$, we use the formula

$$\mu(D) = \int\limits_{\Pr_k(D)} \mu(D(x_1, ...x_{k-1}, x_{k+1}, ..., x_s))dx_1...dx_{k-1}dx_{k+1}...dx_s, \tag{77}$$

where $D(x_1, ...x_{k-1}, x_{k+1}, ..., x_s) = \{x_k : (x_1, x_2, ..., x_s) \in D\}$.

ESTIMATION 1. Here we prove (72) by using (76). During this estimation the set S_ρ is redenoted by G. First we estimate $\mu(G(+1,a))$ for $a = \rho^{1-\alpha}$ by using (76) for $k = 1$ and the following relations

$$\frac{\partial F(x)}{\partial x_1} = l \mid x \mid^{2(l-1)} (2x_1 + O(\rho^{-2\alpha})) \tag{78}$$

$$\frac{\partial F}{\partial x_1} > \rho^{1-\alpha},\ (\frac{\partial F}{\partial x_1})^{-1} \mid grad(F) \mid= \frac{\rho}{\sqrt{\rho^2 - x_2^2 - x_3^2 - ... - x_d^2}} + O(\rho^{-\alpha}), \tag{79}$$

$$\Pr(G(+1,a)) \supset \Pr(A(+1,2a)), \tag{80}$$

where $x \in G(+1,a)$, $A = B(\rho) \cap U^1(3\rho^{\alpha_1}, p)$, $B(\rho) = \{x \in \mathbb{R}^d :\mid x \mid= \rho\}$. The estimations (78) and (79) follow from (47). Now we prove (80). If

$(x_2,...,x_d)\in \Pr_1(A(+1,2a))$, then there exists x_1 such that

$$x_1>2a=2\rho^{1-\alpha},\ x_1^2+x_2^2+...+x_d^2=\rho^2, \mid \sum_{i\geq 1}(2x_ib_i-b_i^2)\mid \geq 3\rho^{\alpha_1} \qquad (81)$$

for all $(b_1,b_2,...,b_d)\in\Gamma(p\rho^\alpha)$. Therefore for $h=\rho^{-\alpha}$, we have

$(x_1+h)^2+x_2^2+...+x_q^2>\rho^2+\rho^{-\alpha},(x_1-h)^2+x_2^2+...+x_q^2<\rho^2-\rho^{-\alpha}$. These inequalities and (24) give

$F(x_1+h,x_2,...,x_d)>\rho^{2l}$, $F(x_1-h,x_2,...,x_d)<\rho^{2l}$. Since F is a continuous function, there exists $y_1\in(x_1-h,x_1+h)$ such that

$$y_1>a, F(y_1,x_2,...,x_d)=\rho^{2l},\quad \mid 2y_1b_1-b_1^2+\sum_{i\geq 2}(2x_ib_i-b_i^2)\mid>2\rho^{\alpha_1}, \qquad (82)$$

because the expression under the absolute value in (82) differ from the expression under the absolute value in (81) by $2(y_1-x_1)b_1$, where $\mid y_1-x_1\mid<h=\rho^\alpha$, $b_1<p\rho^\alpha$, $\mid 2(y_1-x_1)b_1\mid<2p\rho^{2\alpha}<\rho^{\alpha_1}$. The relations in (82) means that $(x_2,...,x_d)\in\Pr G(+1,a)$. Hence (80) is proved. Now (76), (79), and the obvious relation $\mu(\Pr G(+1,a))=O(\rho^{d-1})$ (see the inequalities in (75)) imply that

$$\mu(G(+1,a))=\int\limits_{\Pr(G(+1,a))}\frac{\rho}{\sqrt{\rho^2-x_2^2-x_3^2-...-x_d^2}}dx_2dx_3...dx_d+O(\frac{1}{\rho^\alpha})\mu(B(\rho))$$

$$\geq\int\limits_{\Pr(A(+1,2a))}\frac{\rho}{\sqrt{\rho^2-x_2^2-x_3^2-...-x_d^2}}dx_2dx_3...dx_d-c_{12}\rho^{-\alpha}\mu(B(\rho))$$

$$=\mu(A(+1,2a))-c_{12}\rho^{-\alpha}\mu(B(\rho)).$$

Similarly, $\mu(G(-1,a))\geq\mu(A(-1,2a))-c_{12}\rho^{-\alpha}\mu(B(\rho))$. Now using the inequality $\mu(G)\geq\mu(G(+1,a))+\mu(G(-1,a))$ we get

$\mu(G)\geq\mu(A(-1,2a))+\mu(A(+1,2a))-2c_{12}\rho^{-\alpha}\mu(B(\rho))$. On the other hand it follows from the obvious relation

$\mu(\{x\in B(\rho):-2a\leq x_1\leq 2a\})=O(\rho^{-\alpha})\mu(B(\rho))$ that

$\mu(A(-1,2a))+\mu(A(+1,2a))\geq\mu(A)-c_{13}\rho^{-\alpha}\mu(B(\rho))$. Therefore

$\mu(G)>\mu(A)-c_{14}\rho^{-\alpha}\mu(B(\rho))$. It implies (72), since

$\mu(A))=(1+O(\rho^{-\alpha}))\mu(B(\rho))$ (see (45)).

ESTIMATION 2 Here we prove (73). For this we estimate the measure of the set $S_\rho\cap P_b$ (see (63)) by using (76). During this estimation the set $S_\rho\cap P_b$ is redenoted by G. We choose the coordinate axis so that the direction of b coincides with the direction of $(1,0,0,...,0)$, i.e., $b=(b_1,0,0,...,0)$ and $b_1>0$. It follows from the definitions of S_ρ, P_b and $F(x)$ (see the beginning of this section and (63)) that, if $(x_1,x_2,...,x_d)\in G=S_\rho\cap P_b$, then

$$(x_1^2+x_2^2+...+x_d^2)^l+F_{k_1-1}(x)=\rho^{2l}, \qquad (83)$$

$$((x_1-b_1)^2+x_2^2+x_3^2+...+x_d^2)^l+F_{k_1-1}(x+b)=\rho^{2l}+h, \qquad (84)$$

where $h \in (-3\varepsilon_1, 3\varepsilon_1)$, and by (24), it follows that

$$(x_1^2 + x_2^2 + x_3^2 + ... + x_d^2) = \rho^2 + O(\rho^{-\alpha_1}) \tag{85}$$

$$((x_1 - b_1)^2 + x_2^2 + x_3^2 + ... + x_d^2) = \rho^2 + O(\rho^{-\alpha_1}). \tag{86}$$

Subtracting (85) from (86), we get

$$(2x_1 - b_1)b_1 = O(\rho^{-\alpha_1}). \tag{87}$$

Now (87) and the inequalities in (75) imply

$$\mid b_1 \mid < 2\rho + 3,\ x_1 = \frac{b_1}{2} + O(\rho^{-\alpha_1} b_1^{-1}), \mid x_1^2 - (\frac{b_1}{2})^2 \mid = O(\rho^{-\alpha_1}) \tag{88}$$

Consider two cases. Case 1: $b \in \Gamma_1$, where $\Gamma_1 = \{b \in \Gamma : \mid \rho^2 - (\frac{b_1}{2})^2 \mid < 3d\rho^{-2\alpha}\}$. In this case using $\alpha_1 = 3\alpha$, the last equality in (88), and (85), we obtain

$$x_1^2 = \rho^2 + O(\rho^{-2\alpha}), \mid x_1 \mid = \rho + O(\rho^{-2\alpha-1}),\ x_2^2 + x_3^2 + ... + x_d^2 = O(\rho^{-2\alpha}). \tag{89}$$

Therefore $G \subset G(+1, a) \cup G(-1, a)$, where $a = \rho - \rho^{-1}$. Using (76), the relation $\mu(\mathrm{Pr}_1(G(+1, a)) = O(\rho^{-(d-1)\alpha})$ (see the last equality in (89)) and taking into account that the expression under the integral in (76) for $k = 1$ is equal to $1 + O(\rho^{-\alpha})$ (see (79) and (89)), we get $\mu(G(+1, a)) = O(\rho^{-(d-1)\alpha})$. Similarly, $\mu(G(-1, a)) = O(\rho^{-(d-1)\alpha})$. Thus $\mu(G) = O(\rho^{-(d-1)\alpha})$. Since $\mid \Gamma_1 \mid = O(\rho^{d-1})$, we have

$$\mu(\cup_{b \in \Gamma_1}(S_\rho \cap P_b) = O(\rho^{-(d-1)\alpha + d-1}) = O(\rho^{-\alpha})\mu(B(\rho)). \tag{90}$$

Case 2: $b \notin \Gamma_1$. Then using (88), (85), and $\alpha_1 = 3\alpha$, we get

$$\mid x_1^2 - \rho^2 \mid > 2d\rho^{-2\alpha},\ \sum_{k=2}^{d} x_k^2 > d\rho^{-2\alpha},\ \max_{k \geq 2} \mid x_k \mid > \rho^{-\alpha}. \tag{91}$$

Therefore $G \subset \cup_{k \geq 2}(G(+k, \rho^{-\alpha}) \cup G(-k, \rho^{-\alpha}))$. Now we estimate $\mu(G(+d, \rho^{-\alpha}))$ by using (76). Redenote by D the set $\mathrm{Pr}_d\, G(+d, \rho^{-\alpha})$. If $x \in G(+d, \rho^{-\alpha})$, then according to (85) and (47) the under integral expression in (76) for $k = d$ is $O(\rho^{1+\alpha})$. Therefore the first equality in

$$\mu(D) = O(\varepsilon_1 \mid b \mid^{-1} \rho^{d-2}),\ \mu(G(+d, \rho^{-\alpha})) = O(\rho^{d-1+\alpha} \varepsilon_1 \mid b \mid^{-1}) \tag{92}$$

implies the second equality in (92). To prove the first equality in (92) we use (77) for $s = d-1$ and $k = 1$ and prove the relations $\mu(\mathrm{Pr}_1\, D) = O(\rho^{d-2})$,

$$\mu(D(x_2, x_3, ..., x_{d-1})) < 6\varepsilon_1 \mid b \mid^{-1} \tag{93}$$

for $(x_2, x_3, ..., x_{d-1}) \in \mathrm{Pr}_1\, D$. First relation follows from the inequalities in (75)). So we need to prove (93). If $x_1 \in D(x_2, x_3, ..., x_{d-1})$ then (83) and (84) hold. Subtracting (83) from (84), we get

$$((x_1 - b_1)^2 + x_2^2 + x_3^2 + ... + x_d^2)^l - ((x_1^2 + x_2^2 + x_3^2 + ... + x_d^2)^l$$

$$+F_{k_1-1}(x-b)-F_{k_1-1}(x)=h, \tag{94}$$

where $x_2, x_3, ..., x_{d-1}$ are fixed . Hence we have two equations (83) and (94) with respect to two unknown x_1 and x_d. Using (47), the implicit function theorem, and the inequalities $\mid x_d \mid> \rho^{-\alpha}$, $\alpha_1 > 2\alpha$ from (83), we obtain

$$x_d = f(x_1),\ \frac{df}{dx_1} = -\frac{2x_1 + O(\rho^{-2\alpha_1+\alpha})}{2x_d + O(\rho^{-2\alpha_1+\alpha})} \tag{95}$$

Substituting $f(x_1)$ for x_d in (94), we get

$$((x_1-b_1)^2 + x_2^2 + x_3^2 + ... + f^2(x_1))^l - ((x_1^2 + x_2^2 + x_3^2 + ... + f^2(x_1))^l +$$

$$F_{k_1-1}(x_1-b_1, x_2, ..., x_{d-1}, f(x_1)) - F_{k_1-1}(x_1, ..., x_{d-1}, f) = h \tag{96}$$

Using (47), (95), the first equality in (88), and $\mid x_d \mid> \rho^{-\alpha}$ we see that the derivative (w.r.t. x_1) of the right-hand side $a_l(x)$ of (96) is

$$\frac{\partial a_l(x)}{\partial x_1} = l \mid x-b \mid^{2(l-1)} (2(x_1-b_1) + 2f(x_1)f^{'}(x_1)) \tag{97}$$

$$-l \mid x \mid^{2(l-1)} (2x_1 + 2f(x_1)f^{'}(x_1)) + O(\rho^{-2\alpha_1+\alpha})(1 - \tfrac{x_1+O(\rho^{-2\alpha_1+\alpha})}{x_d+O(\rho^{-2\alpha_1+\alpha})}).$$

If $l=1$, then using the first equality in (88) and $\mid x_d \mid> \rho^{-\alpha}$, we get

$$\mid \frac{\partial a_l(x)}{\partial x_1} \mid> b_1 =\mid b \mid \tag{98}$$

If $l>1$, then it follows from (83), (84), and (24) that

$$\mid x \mid^{2l} = \rho^{2l}(1+O(\rho^{-2l-\alpha_1})),\ \ \mid x-b \mid^{2l} = \rho^{2l}(1+O(\rho^{-2l-\alpha_1}))$$

$$\mid x \mid^{2(l-1)} = \rho^{2(l-1)}(1+O(\rho^{-2l-\alpha_1}))^{\frac{l-1}{l}} = \rho^{2(l-1)} + O(\rho^{-2-\alpha_1})$$
$$\mid x-b \mid^{2(l-1)} = \rho^{2(l-1)}(1+O(\rho^{-2l-\alpha_1}))^{\frac{l-1}{l}} = \rho^{2(l-1)} + O(\rho^{-2-\alpha_1})$$

Using these in (97) and arguing as in proof of (98) for $l=1$, we get the proof of (98) for $l>1$. Thus, in any case (98) holds. Therefore from (96), by using the implicit function theorem, we get $\mid \frac{dx_1}{dh} \mid< \frac{1}{|b|}$. This inequality and relation $h \in (-3\varepsilon_1, 3\varepsilon_1)$ imply (93). Hence (92) is proved. In the same way we get the same estimation for $G(+k, \rho^{-\alpha})$ and $G(-k, \rho^{-\alpha})$ for $k \geq 2$. Thus

$\mu(S_\rho \cap P_b) = O(\rho^{d-1+\alpha}\varepsilon_1 \mid b \mid^{-1})$ for $b \notin \Gamma_1$. Since $\varepsilon_1 = \rho^{-d-2\alpha}$, $\mid b \mid< 2\rho+3$ (see (88)), using that the number of the vectors of Γ satisfying $\mid b \mid< 2\rho+3$ is $O(\rho^d)$, we get $\mu(\cup_{b\notin\Gamma_1}(S_\rho \cap P_b)) = O(\rho^{2d-1+\alpha}\varepsilon_1) = O(\rho^{-\alpha})\mu(B(\rho))$. Therefore (90) and (72) imply the proof of (73).

ESTIMATION 3. Here we prove (74). Denote $U_{2\varepsilon}(A_{k,j}(\gamma_1, \gamma_2, ..., \gamma_k))$ by G, where $\gamma_1, \gamma_2, ..., \gamma_k \in \Gamma(p\rho^\alpha), k \leq d-1$, and $A_{k,j}$ is defined at the beginning of this section. We turn the coordinate axis so that

$Span\{\gamma_1, \gamma_2, ..., \gamma_k\} = \{x = (x_1, x_2, ..., x_k, 0, 0, ..., 0) : x_1, x_2, ..., x_k \in \mathbb{R}\}$. Then by (35), we have $x_n = O(\rho^{\alpha_k+(k-1)\alpha})$ for $n \leq k$, $x \in G$. This, (75), and $\alpha_k + (k-1)\alpha < 1$ (see the sixth inequality in (15)) give $G \subset (\cup_{i>k}(G(+i, \rho d^{-1}) \cup G(-i, \rho d^{-1}))$, $\mu(\Pr_i(G(+i, \rho d^{-1}))) = O(\rho^{k(\alpha_k+(k-1)\alpha)+(d-1-k)})$ for $i > k$. Now using this and (77) for $s = d$, we prove that

$$\mu(G(+i, \rho d^{-1})) = O(\varepsilon\rho^{k(\alpha_k+(k-1)\alpha)+(d-1-k)}), \forall i > k. \tag{99}$$

For this we redenote by D the set $G(+i, \rho d^{-1})$ and prove that

$$\mu((D(x_1, x_2, ...x_{i-1}, x_{i+1}, ...x_d)) \leq (42d^2 + 4)\varepsilon \tag{100}$$

for $(x_1, x_2, ...x_{i-1}, x_{i+1}, ...x_d) \in \Pr_i(D)$ and $i > k$. To prove (100) it is sufficient to show that if both $x = (x_1, x_2, ..., x_i, ...x_d)$ and $x^{'} = (x_1, x_2, ..., x_i^{'}, ..., x_d)$ are in D, then $\mid x_i - x_i^{'} \mid \leq (42d^2 + 4)\varepsilon$. Assume the converse. Then $\mid x_i - x_i^{'} \mid > (42d^2 + 4)\varepsilon$. Without loss of generality it can be assumed that $x_i^{'} > x_i$. So $x_i^{'} > x_i > \rho d^{-1}$ (see definition of D). Since x and $x^{'}$ lie in the 2ε neighborhood of $A_{k,j}$, there exist points $a, a^{'}$ in $A_{k,j}$ such that $\mid x - a \mid < 2\varepsilon$, $\mid x^{'} - a^{'} \mid < 2\varepsilon$. It follows from the definitions of the points x, $x^{'}, a$, $a^{'}$ that the following inequalities hold:

$$\rho d^{-1} - 2\varepsilon < a_i < a_i^{'},\ a_i^{'} - a_i > 42d^2\varepsilon, \tag{101}$$
$$(a_i^{'})^2 - (a_i)^2 > 2(\rho d^{-1} - 2\varepsilon)(a_i^{'} - a_i),$$
$$\mid\mid a_s \mid - \mid a_s^{'} \mid\mid < 4\varepsilon, \forall s \neq i.$$

On the other hand the inequalities in (75) hold for the points of $A_{k,j}$, that is, we have $\mid a_s \mid < \rho + 1, \mid a_s^{'} \mid < \rho + 1$. Therefore these inequalities and the inequalities in (101) imply $\mid\mid a_s \mid^2 - \mid a_s^{'} \mid^2 \mid < 12\rho\varepsilon$ for $s \neq i$, and hence $\sum_{s \neq i} \mid\mid a_s \mid^2 - \mid a_s^{'} \mid^2 \mid < 12d\rho\varepsilon < \frac{2}{7}\rho d^{-1}(a_i^{'} - a_i)$,

$$\mid\mid a \mid^2 - \mid a^{'} \mid^2 \mid > \frac{3}{2}\rho d^{-1} \mid a_i^{'} - a_i \mid . \tag{102}$$

Moreover, using mean value theorem and the relations $\mid a \mid = \rho + O(1), \mid a^{'} \mid = \rho + O(1)$ (see (75)), we get

$$\mid a \mid^{2l} - \mid a^{'} \mid^{2l} = l(\rho + O(1))^{2(l-1)})(\mid a \mid^2 - \mid a^{'} \mid^2) \tag{103}$$

Let $r_i(x) = \lambda_i(x) - \mid x \mid^{2l}$, where $\lambda_i(x)$ is the eigenvalues of the matrix $C(x)$ defined in Theorem 2. Hence $r_1(x) \leq r_2(x) \leq ... \leq r_{b_k}(x)$ are the eigenvalues of the matrix $C^{'}(x)$, where $C^{'}(x) = C(x) - \mid x \mid^{2l} I$. By definition of $C^{'}(x)$ only diagonal elements of the matrix $C^{'}(x)$ depend on x and they are $\mid x - d_i \mid^{2l} - \mid x \mid^{2l}$, where $d_i = h_i + t - \gamma - t \in B_k + \Gamma(p_1\rho^{\alpha})$. Clearly,

$$\mid d_i \mid < \rho^{\frac{1}{2}\alpha_d},\ \ \mid r_j(a^{'}) - r_j(a) \mid \leq \parallel C^{'}(a^{'}) - C^{'}(a) \parallel = \max_i \mid a_{i,i} \mid, \tag{104}$$

where $C^{'}(a^{'})-C^{'}(a)=(a_{i,j})$, $a_{i,i}=\mid a\mid^{2l}-\mid a-d_i\mid^{2l}-\mid a^{'}\mid^{2l}+\mid a^{'}-d_i\mid^{2l}$, and $a_{i,j}=0$ for $i\neq j$, that is, $C^{'}(x)-C^{'}(x^{'})$ is a diagonal matrix. Now we estimate $\mid a_{i,i}\mid$. Using mean value theorem and the relations $\mid a\mid=\rho+O(1)$,

$\mid a^{'}\mid=\rho+O(1)$, $\mid a-d_i\mid=\rho+O(\rho^{\frac{1}{2}\alpha_d})$, $\mid a^{'}-d_i\mid=\rho+O(\rho^{\frac{1}{2}\alpha_d})$, we obtain

$$a_{i,i}=l(\rho+O(\rho^{\frac{1}{2}\alpha_d}))^{2(l-1)}(\mid a\mid^2-\mid a^{'}\mid^2)-$$

$$l(\rho+O(\rho^{\frac{1}{2}\alpha_d}))^{2(l-1)}(\mid a-d_i\mid^2-\mid a^{'}-d_i\mid^2)=$$

$$l(\rho+O(\rho^{\frac{1}{2}\alpha_d}))^{2(l-1)}(\mid a\mid^2-\mid a-d_i\mid^2-\mid a^{'}\mid^2+\mid a^{'}-d_i\mid^2)+$$

$$O(\rho^{\frac{1}{2}\alpha_d+2l-3})(\mid a\mid^2-\mid a^{'}\mid^2).$$

Since $\mid a\mid^2-\mid a-d_i\mid^2-\mid a^{'}\mid^2+\mid a^{'}-d_i\mid^2=2(a-a^{'},d_i)$, we have (see (104)) $\mid r_j(a)-r_j(a^{'})\mid\leq 3l\rho^{\frac{1}{2}\alpha_d+2l-2}\mid a-a^{'}\mid -c_{15}\rho^{\frac{1}{2}\alpha_d+2l-3}\mid\mid a\mid^2-\mid a^{'}\mid^2\mid$. Therefore using $\frac{1}{2}\alpha_d<1$, (103), (102), (101) and $\lambda_i(x)=r_i(x)+\mid x\mid^{2l}$, we obtain

$\mid\lambda_j(a)-\lambda_j(a^{'})\mid\geq\mid\mid a\mid^{2l}-\mid a^{'}\mid^{2l}\mid-\mid r_j(a)-r_j(a^{'})\mid>$

$l\rho^{2l-1}d^{-1}\mid a_i^{'}-a_i\mid>l\rho^{2l-1}42d\varepsilon>6\varepsilon_1$, which contradicts the fact that both $\lambda_j(a)$ and $\lambda_j(a^{'})$ lie in $(\rho^2-3\varepsilon_1,\rho^2+3\varepsilon_1)$ (see the definition of $A_{k,j}$). Thus (100), hence (99) is proved. In the same way we get the same formula for $G(-i,\frac{\rho}{d})$. So $\mu(U_{2\varepsilon}(A_{k,j}(\gamma_1,\gamma_2,...,\gamma_k)))=O(\varepsilon\rho^{k(\alpha_k+(k-1)\alpha)+d-1-k})$, where

$j=1,2,...,b_k(\gamma_1,\gamma_2,...,\gamma_k)$, and $\gamma_1,\gamma_2,...,\gamma_k\in\Gamma(p\rho^\alpha)$. From this using that $b_k=O(\rho^{d\alpha+\frac{k}{2}\alpha_{k+1}})$ (see (43)) and the number of the vectors $(\gamma_1,\gamma_2,...,\gamma_k)$ for $\gamma_1,\gamma_2,...,\gamma_k\in\Gamma(p\rho^\alpha)$ is $O(\rho^{dk\alpha})$, we obtain (74) if

$d\alpha+\frac{k}{2}\alpha_{k+1}+dk\alpha+k(\alpha_k+(k-1)\alpha)+d-1-k\leq d-1-\alpha$ or

$$(d+1)\alpha+\frac{k}{2}\alpha_{k+1}+dk\alpha+k(\alpha_k+(k-1)\alpha)\leq k \tag{105}$$

for $1\leq k\leq d-1$. Dividing both side of (105) by $k\alpha$ and using $\alpha_k=3^k\alpha$, $\alpha=\frac{1}{m}$, $m=3^d+d+2$ (see the end of the introduction) we see that (105) is equivalent to $\frac{d+1}{k}+\frac{3^{k+1}}{2}+3^k+k-1\leq 3^d+2$

The left-hand side of this inequality gets its maximum value at $k=d-1$. Therefore we need to show that $\frac{d+1}{d-1}+\frac{5}{6}3^d+d\leq 3^d+4$, which follows from the obvious inequalities $\frac{d+1}{d-1}\leq 3, d<\frac{1}{6}3^d+1$ for $d\geq 2$.

ESTIMATION 4. Here we prove (69). During this estimation we denote by G the set $S_\rho^{'}\cap U_\varepsilon(Tr(A(\rho))$. Since $V_\rho=S_\rho^{'}\backslash G$ and (73) holds, it is enough to prove that $\mu(G)=O(\rho^{-\alpha})\mu(B(\rho))$. For this we use (75) and prove $\mu(G(+i,\rho d^{-1}))=O(\rho^{-\alpha})\mu(B(\rho))$ for $i=1,2,...,d$ by using (76) (the same estimation for $G(-i,\rho d^{-1})$ can be proved in the same way). By (47), if $x\in G(+i,\rho d^{-1})$, then the under integral expression in (76) for $k=i$ and $a=\rho d^{-1}$ is less than $d+1$. Therefore it is sufficient to prove

$$\mu(\Pr(G(+i,\rho d^{-1}))=O(\rho^{-\alpha})\mu(B(\rho)) \tag{106}$$

Clearly, if $(x_1, x_2, ...x_{i-1}, x_{i+1}, ...x_d) \in \Pr_i(G(+i, \rho d^{-1}))$, then $\mu(U_\varepsilon(G)(x_1, x_2, ...x_{i-1}, x_{i+1}, ...x_d)) \geq 2\varepsilon$ and by (77), it follows that

$$\mu(U_\varepsilon(G)) \geq 2\varepsilon\mu(\Pr(G(+i, \rho d^{-1})). \tag{107}$$

Hence to prove (106) we need to estimate $\mu(U_\varepsilon(G))$. For this we prove that

$$U_\varepsilon(G) \subset U_\varepsilon(S_\rho^{'}), U_\varepsilon(G) \subset U_{2\varepsilon}(Tr(A(\rho))), U_\varepsilon(G) \subset Tr(U_{2\varepsilon}(A(\rho))). \tag{108}$$

The first and second inclusions follow from $G \subset S_\rho^{'}$ and $G \subset U_\varepsilon(Tr(A(\rho)))$ respectively (see definition of G). Now we prove the third inclusion in (108). If $x \in U_\varepsilon(G)$, then by the second inclusion of (108) there exists b such that $b \in Tr(A(\rho))$, $\mid x - b \mid < 2\varepsilon$. Then by the definition of $Tr(A(\rho))$ there are $\gamma \in \Gamma$ and $c \in A(\rho)$ such that $b = \gamma + c$. Therefore $\mid x - \gamma - c \mid = \mid x - b \mid < 2\varepsilon$, $x - \gamma \in U_{2\varepsilon}(c) \subset U_{2\varepsilon}(A(\rho))$. This together with $x \in U_\varepsilon(G) \subset U_\varepsilon(S_\rho^{'})$ (see the first inclusion of (108)) give $x \in Tr(U_{2\varepsilon}(A(\rho)))$, i.e., the third inclusion in (108) is proved. The third inclusion, Lemma 2(c), and (74) imply that $\mu(U_\varepsilon(G)) = O(\rho^{-\alpha})\mu(B(\rho))\varepsilon$. This and (107) imply the proof of (106)$\diamondsuit$

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